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Fast and Slow Convergence to Equilibrium for Maxwellian Molecules via Wild Sums

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We consider the spatially homogeneous Boltzmann equation for Maxwellian molecules and general finite energy initial data: positive Borel measures with finite moments up to order 2. We show that the coefficients in the Wild sum converge strongly to the equilibrium, and quantitatively estimate the rate. We show that this depends on the initial data F essentially only through on the behavior near $r=0$ of the function $J_F(r)=\int_{|v|>1/r}|v|^2\,dF(v)$. These estimates on the terms in the Wild sum yield a quantitative estimate, in the strongest physical norm, on the rate at which the solution converges to equilibrium, as well as a global stability estimate. We show that our upper bounds are qualitatively sharp by producing examples of solutions for which the convergence is as slow as permitted by our bounds. These are the first examples of solutions of the Boltzmann equation that converge to equilibrium more slowly than exponentially.

KEY WORDS: Boltzmann equation; Maxwellian molecules; Wild sum; global stability.

1. INTRODUCTION

1.1. The Boltzmann Equation in $L^1(R^3)$

The spatially homogeneous Boltzmann equation describes the time evolution of the velocity density function $f_t(v)\equiv f(v,t)$ for a dilute gas composed

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of identical particles. Hence f_t should be a non-negative integrable function on $\mathbf{R}^3$, and the equation itself is

$$\frac{\partial}{\partial t} f_t(v) = Q(f_t, f_t)(v), \qquad (v, t) \in \mathbf{R}^3 \times [0, \infty) \tag{1.1}$$

where and Q is the *collision integral operator* given by

$$Q(f, g)(v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \sigma)[f(v')\, g(v'_*) - f(v)\, g(v_*)]\, d\sigma\, dv_*.$$

Here, v' and v'_* denote the *post-collisional velocites*, and they must obey the conservation of momentum and kinetic energy:

$$v' + v'_* = v + v_*, \qquad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$

All such pairs of vectors may be parameterized by (unit) vectors $\sigma \in \mathbf{S}^2$. One particularly useful parameterization is

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \qquad \sigma \in \mathbf{S}^2. \tag{1.2}$$

The function B is the *collision kernel*, which is a non negative Borel function of $|v - v_*|$ and $\langle v - v_*, \sigma \rangle$ only.

The results in this paper will all be obtained under two conditions on B. The first is that we consider so-called *Maxwellian molecules*, meaning that the kernel B depends only on $\langle v - v_*, \sigma \rangle / |v - v_*|$:

$$B(v - v_*, \sigma) = B\left(\left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle\right). \tag{1.3}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{R}^3$. For background information on the physical model, see, e.g., refs. 13 and 29. For later convenience, we define $B(t) = 0$ for $|t| > 1$. The second is the so-called *angular cutoff condition*, which is

$$\int_0^\pi B(\cos(\theta)) \sin(\theta)\, d\theta < \infty. \tag{1.4}$$

These two conditions have the following consequences. Under the angular cutoff condition (1.4), the collision integral operator $Q(f, g)$ can be split into its so-called *gain* and *loss* terms:

$$Q(f, g)(v) = Q^+(f, g)(v) - Q^-(f, g)(v) \tag{1.5}$$

where

$$Q^{+}(f, g)(v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \sigma)\, f(v')\, g(v'_*)\, d\sigma\, dv_*,$$

$$Q^{-}(f, g)(v) = f(v) \int_{\mathbf{R}^3} \left[\int_{\mathbf{S}^2} B(v - v_*, \sigma)\, d\sigma \right] g(v_*)\, dv_*.$$

Without the cutoff condition, cancellations in $Q(f, g)$ are crucial, and solutions to (1.1) have only been shown to exist and studied in a certain weak form (refs. 4, 17, 34, 3, and 2). The cutoff condition facilitates the construction of strong solutions that can be studied in greater detail.

This is particularly true in the case that B satisfies the Mawellian molecule condition. Then there is a constructive method, due to Wild,[33] of solving (1.1) for any non-negative initial datum $f_0 = F \in L^1(\mathbf{R}^3)$. Wild's method gives the (unique) solution in the form

$$f_t = \sum_{n=1}^{\infty} e^{-at}(1 - e^{-at})^{n-1}\, Q_n^{+}(F)$$

where $a = 2\pi \int_0^{\pi} B(\cos(\theta)) \sin(\theta)\, d\theta\, \|F\|_{L^1}$ and the $Q_n^{+}(F)$ are certain recursively defined integral expressions in F. We describe these below in great detail, but before that, we wish to focus attention on the following point: Wild's formula allows one to rephrase questions about the evolution $t \mapsto f_t$ as questions about the development of the sequence $n \mapsto Q_n^{+}(F)$. As we shall see below, they are often much more amenable in this form. This is particularly true of questions concerning the trend to equilibrium.

As is well known, the only equilibrium (or steady-state) solutions of (1.1) in L^1 are the *Maxwellians*. This is the family of non-negative integral functions on $\mathbf{R}^3$ paramterized by $\rho \in \mathbf{R}_+$, $u \in \mathbf{R}^3$ and $T \in \mathbf{R}_+$ by

$$M_{\rho, u, T}(v) = \rho(2\pi T)^{-3/2}\, e^{-|v-u|^2/2T}.$$

Given a non-negative function $F \in L^1_2(\mathbf{R}^3)$, define M_F to be the Maxwellian with parameters (ρ, u, T) where

$$\int_{\mathbf{R}^3} (1, v, |v|^2)\, F(v)\, dv = (\rho, \rho u, 3\rho T).$$

For instance under the moment condition $\int_{\mathbf{R}^3} (1, v, |v|^2)\, F(v)\, dv = (1, 0, 3)$ the corresponding Maxwellian M_F is

$$M_F(v) = M(v) := \left(\frac{1}{2\pi}\right)^{3/2} e^{-|v|^2/2}.$$

It is also convenient to establish the notations

$$\rho_F = \int_{\mathbf{R}^3} F(v)\, dv, \quad u_F = \frac{1}{\rho_F} \int_{\mathbf{R}^3} vF(v)\, dv \qquad \text{and}$$
$$T_F = \frac{1}{3\rho_F} \int_{\mathbf{R}^3} |v - u_F|^2\, F(v)\, dv. \tag{1.6}$$

We shall show here that in broadest physical generality,

$$\lim_{n\to\infty} Q_n^+(F) = M_F$$

in the strongest physical norms, and shall obtain precise rate information on the convergence. This in turn shall provide us with precise information concerning how the rate at which f_t approaches M_F depends on F and B.

We now turn to some technical matters and notational conventions required for a clear statement of our main results.

Define $L^1_s(\mathbf{R}^3)$ to be the weighted L^1 space:

$$L^1_s(\mathbf{R}^3) = \left\{ f \mid \|f\|_{L^1_s} := \int_{\mathbf{R}^3} (1+|v|^2)^{s/2}\, |f(v)|\, dv < \infty \right\}, \qquad s \geqslant 0$$

where f are real or complex valued measurable function on $\mathbf{R}^3$.

It is well known that for any initial datum $0 \leqslant F \in L^1_2(\mathbf{R}^3)$, the solution to (1.1) with (1.3)–(1.4) in the class $C^1([0, \infty); L^1_2(\mathbf{R}^3))$ exists uniquely and conserves the mass, momentum (or mean velocity) and energy. In fact, even without the cutoff assumption, global existence, uniqueness, and the conservation laws have been proven in Toscani and Villani[(27)] for all initial data having finite mass and finite energy.

Because of the integrability in (1.4), we can assume (after rescaling the time variable) that the total integral of the kernel B on $\mathbf{S}^2$ is one:

$$\int_{\mathbf{S}^2} B(\langle \zeta, \sigma \rangle)\, d\sigma = 2\pi \int_0^\pi B(\cos(\theta))\sin(\theta)\, d\theta = 2\pi\, \|B\|_{L^1[-1,1]} = 1, \qquad \zeta \in \mathbf{S}^2. \tag{1.7}$$

Under this condition, for f and g non-negative integrable functions,

$$\int_{\mathbf{R}^3} Q^+(f, g)\, dv = \|f\|_{L^1}\, \|g\|_{L^1}.$$

Hence, if f and g are both probability densities, so is $Q^+(f, g)$. As a consequence of (1.2), changing σ to $-\sigma$ does not change the product

$f(v')\, f(v'_*)$. So it is usual to assume that the angular function $B(\cdot)$ is even: $B(-t) = B(t)$, $t \in [-1, 1]$. This then implies that

$$Q^+(f, g)(v) = Q^+(g, f)(v).$$

Following Wild, we now define the *Wild convolution* $f \circ g$ of two integrable functions f and g on $\mathbf{R}^3$ by

$$f \circ g = Q^+(f, g).$$

By what has been said above, under the condition that B is even, the Wild convolution is commutative. However, it is not associative under any natural condition on B. *Throughout this paper, unless otherwise stated, we always assume that the kernel* $B(\cdot)$ *is a nonnegative even function in* $L^1(\mathbf{R})$ *with* supp $B \subset [-1, 1]$ *and satisfies the normalization* (1.7).

Using this notation, and under our assumptions, we can rewrite (1.1) in the form

$$\frac{\partial}{\partial t} f_t(v) = (f_t \circ f_t)(v) - \rho_F\, f_t(v), \qquad f_t|_{t=0} = F. \tag{1.8}$$

This may be solved by iteration in a standard way, producing a solution in the form of a so-called *Wild sum*[33] (for instance for $\rho_F = 1$):

$$f_t = \sum_{n=1}^{\infty} e^{-t}(1 - e^{-t})^{n-1}\, Q_n^+(F) \tag{1.9}$$

where

$$Q_1^+(F) = F$$

and

$$Q_n^+(F) = \frac{1}{n-1} \sum_{k=1}^{n-1} Q_k^+(F) \circ Q_{n-k}^+(F), \qquad n \geqslant 2. \tag{1.10}$$

One can check this by showing that the right hand side of (1.9) is also an energy conserving solution of the Boltzmann equation with the same initial datum F, and since such solutions are unique, it is *the* solution.

It is known that for all non-negative $F \in L^1_2(\mathbf{R}^3)$, the solution of (1.8) satsifies

$$\lim_{t \to \infty} f_t = M_F. \tag{1.11}$$

McKean observed an analogy between this convergence and the convergence to a Gaussian distribution in the central limit theorem.[22] He proposed that there should be a *central limit theorem for Maxwellian molecules* asserting that (for $\rho_F = 1$)

$$\lim_{n\to\infty} Q_n^+(F) = M_F. \tag{1.12}$$

Under certain smoothness and moment assumptions on F, this was done in ref. 12, and quantitative bounds were obtained on the rate of convergence in (1.12). Here we take up the question of the rate of convergence in (1.12) in full generality, making no smoothness assumptions on F, and requiring no moment conditions other than finite energy. In fact, not only do we make no smoothness hypothesis on F, we do not even require F to be a function. Our arguments all work assuming only that F is a positive Borel measure with finite second moments. After explaining what it means to solve the Boltzmann equation in this setting, we shall finally state our results.

1.2. Distributional Solutions

In view of statistical physics, initial data to the Boltzmann equation are best chosen from the largest class, say the positive, finite Borel measures on $\mathbf{R}^3$. For the Maxwellian model under consideration, the corresponding solutions, called distributional solutions, give no more essential difficulty than the L^1-solutions in dealing with existence, uniqueness, convergence to equilibrium, stability, etc. Let us introduce some classes of measures.

$$\mathscr{B}_0(\mathbf{R}^3) := \text{finite Borel measures on } \mathbf{R}^3$$

$$\mathscr{B}_s(\mathbf{R}^3) := \left\{ \mu \in \mathscr{B}_0(\mathbf{R}^3) \,\middle|\, \int_{\mathbf{R}^3} (1+|v|^2)^{s/2}\, d\,|\mu|\,(v) < \infty \right\}, \qquad s \geqslant 0$$

$$P_2(\mathbf{R}^3) := \left\{ F \in \mathscr{B}_2(\mathbf{R}^3) \,\middle|\, F \geqslant 0, \int_{\mathbf{R}^3} dF(v) = 1 \right\},$$

$$P_2(\mathbf{R}^3; v_0, T) := \left\{ F \in P_2(\mathbf{R}^3) \,\middle|\, \int_{\mathbf{R}^3} v\, dF(v) = v_0, \tfrac{1}{3} \int_{\mathbf{R}^3} |v - v_0|^2\, dF(v) = T \right\}.$$

Let $\mu \in \mathscr{B}_s(\mathbf{R}^3)$, $s \geqslant 0$. Then μ defines a bounded linear functional on $C_b(\mathbf{R}^3) := C(\mathbf{R}^3) \cap L^\infty(\mathbf{R}^3)$ through

$$\langle \mu, \phi \rangle_s := \int_{\mathbf{R}^3} \phi(v)(1+|v|^2)^{s/2}\, d\mu(v) \qquad \forall \phi \in C_b(\mathbf{R}^3). \tag{1.13}$$

The corresponding norm $\|\cdot\|_{\mathscr{B}_s}$ is given by

$$\|\mu\|_{\mathscr{B}_s} = \sup\{|\langle \mu, \phi \rangle_s| \mid \phi \in C_b(\mathbf{R}^3), \|\phi\|_{L^\infty} \leqslant 1\} = \int_{\mathbf{R}^3} (1+|v|^2)^{s/2}\, d\,|\mu|\,(v). \tag{1.14}$$

For $s=0$, $\|\mu\|_{\mathscr{B}_0}$ is the total variation of the measure μ. Let C_c^∞ denote the class of C^∞-functions with compact supports. It is easily seen that

$$\|\mu\|_{\mathscr{B}_s} = \sup\{|\langle \mu, \phi \rangle_s| \mid \phi \in C_c^\infty(\mathbf{R}^3), \|\phi\|_{L^\infty} \leqslant 1\}. \tag{1.15}$$

Of course if the measure μ is also absolutely continuous with respect to the Lebesgue measure dv, i.e., $d\mu(v) = g(v)\, dv$ for some $g \in L^1_s(\mathbf{R}^3)$, then $\|\mu\|_{\mathscr{B}_s} = \|g\|_{L^1_s}$.

The following relation (see, e.g., Chapter 1 of ref. 35) is a fundamental property of (1.2): For any $f, g \in L^1(\mathbf{R}^3)$, $\phi \in L^\infty(\mathbf{R}^3)$ and $\chi \in L^\infty[0, \infty)$,

$$\int_{\mathbf{R}^3} \phi(v) \left[\iint_{\mathbf{R}^3 \times \mathbf{S}^2} B\left(\left\langle \frac{v-v_*}{|v-v_*|}, \sigma \right\rangle\right) \chi(|v-v_*|)\, f(v')\, g(v'_*)\, d\sigma\, dv_* \right] dv$$
$$= \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \left[\int_{\mathbf{S}^2} B\left(\left\langle \frac{v-v_*}{|v-v_*|}, \sigma \right\rangle\right) \phi(v')\, d\sigma \right] \chi(|v-v_*|)\, f(v)\, g(v_*)\, dv\, dv_*. \tag{1.16}$$

Since a measure may have a singular part, a convention should be made for the collision integral: For any locally bounded Borel function ϕ on $\mathbf{R}^3$, we define

$$L_B[\phi](v, v_*) := \int_{\mathbf{S}^2} B\left(\left\langle \frac{v-v_*}{|v-v_*|}, \sigma \right\rangle\right) \phi(v')\, d\sigma = \phi(v) \qquad \text{for} \quad v = v_*.$$

Under this convention, if ϕ is continuous on $\mathbf{R}^3$, then $L_B[\phi]$ is continuous on $\mathbf{R}^3 \times \mathbf{R}^3$. (To prove this one can assume first that $B(\cdot)$ is smooth, then use standard L^1-approximation).

Now take $F, G \in \mathscr{B}_s(\mathbf{R}^3)$. Let $\chi(r)$ be a bounded Borel function on $[0, \infty)$. Let $Q^+_\chi(\cdot, \cdot)$ be a collision operator with the kernel $B(\langle z/|z|, \sigma \rangle)\, \chi(|z|)$, $z = v - v_*$. Then, using (1.16), the collision integral $Q^+_\chi(F, G)$ is defined to be the Borel measure specified through the Riesz representation theorem by

$$\int_{\mathbf{R}^3} \phi(v)\, dQ^+_\chi(F, G)(v) = \iint_{\mathbf{R}^3 \times \mathbf{R}^3} (L_B[\phi](v, v_*)\, \chi(|v-v_*|))\, dF(v)\, dG(v_*) \tag{1.17}$$

for all Borel function ϕ satisfying $\sup_{v\in\mathbf{R}^3} |\phi(v)|\,(1+|v|^2)^{-s/2} < \infty$. For $\chi = 1$, we simply write $Q^+(F, G)$ in place of $Q^+_\chi(F, G)$. As in L^1, we denote

$$F \circ G := Q^+(F, G)$$

and $F \circ G$ is again called the Wild convolution of $F, G \in \mathscr{B}_s(\mathbf{R}^3)$, and when F and G both belong to $P_2(\mathbf{R}^3; v_0, T)$, so does $F \circ G$. Under our assumptions on B, it is still commutative, and is still not associative.

Among the other useful properties of the Wild convolution is the following continuity property: if $F_1, F_2, G_1, G_2 \in \mathscr{B}_s(\mathbf{R}^3)$, $s \geqslant 0$, then

$$\|F_1 \circ G_1 - F_2 \circ G_2\|_{\mathscr{B}_s} \leqslant \|F_1 - F_2\|_{\mathscr{B}_s}\, \|G_1\|_{\mathscr{B}_s} + \|F_2\|_{\mathscr{B}_s}\, \|G_1 - G_2\|_{\mathscr{B}_s}.$$

When considering distributional solutions to the Boltzmann equation, it is usual to write it in weak form; i.e., to require that $t \mapsto f_t$ be continuous into $\mathscr{B}_2(\mathbf{R}^3, \|\cdot\|_{\mathscr{B}_2})$, that $f_0 = F$, and that

$$\frac{d}{dt}\langle f_t, \phi\rangle_2 = \langle f_t \circ f_t, \phi\rangle_2 - \|F\|_{\mathscr{B}_0}\, \langle f_t, \phi\rangle_2 \qquad \forall \phi \in C_b(\mathbf{R}^3), \quad t \geqslant 0.$$

As in the case of L^1-solutions, the existence, uniqueness and the conservation laws; i.e.,

$$\int_{\mathbf{R}^3} (1, v, |v|^2)\, df_t(v) = \int_{\mathbf{R}^3} (1, v, |v|^2)\, dF(v), \qquad t \geqslant 0 \tag{1.18}$$

have been proven to hold for distributional solutions of Eq. (1.1) for Maxwellian model[27] for all initial data in $\mathscr{B}_2(\mathbf{R}^3)$.

Interpreting integrals of continuous measure valued functions in the Bochner sense, this can be recast as the equivalent integral equation

$$f_t = F + \int_0^t (f_\tau \circ f_\tau)\, d\tau - \|F\|_{\mathscr{B}_0} \int_0^t f_\tau\, d\tau, \qquad t \geqslant 0. \tag{1.19}$$

with initial data $0 \leqslant F \in \mathscr{B}_2(\mathbf{R}^3)$. Here, it is natural to require that the solution $0 \leqslant f_t \in C([0, \infty); \mathscr{B}_2(\mathbf{R}^3, \|\cdot\|_{\mathscr{B}_2}))$ and this then implies that $f_t \in C^1([0, \infty); \mathscr{B}_2(\mathbf{R}^3, \|\cdot\|_{\mathscr{B}_2}))$. The Wild expansion (1.9)–(1.10) also holds for distribution solutions with $\|F\|_{\mathscr{B}_0} = 1$.

Equilibria in the distributional setting are solutions of the following equation:

$$F \circ F = \|F\|_{\mathscr{B}_0}\, F, \qquad 0 \leqslant F \in \mathscr{B}_2(\mathbf{R}^3), \quad \|F\|_{\mathscr{B}_0} \neq 0.$$

In addition to the Maxwellian densities, there is a new class of equilibria: The Dirac measures. Clearly, these are limits of the Maxwellian equilibria, and it is easily checked that Maxwellian distributions and Dirac measures are the only equilibria. (This will also be an easy consequence of the results proved below).

As one final convention, throughout this paper, our notation does not distinguish between Maxwellian function $M(v)$ and Maxwellian distribution $M(v)\,dv$, and we extend the definitions (1.6) for ρ_F, u_F and T_F, as well as M_F, to distributions in the obvious way.

1.3. Main Results

Our main results concern rates of convergence in $\lim_{n\to\infty} Q_n^+(F) = M_F$ and $\lim_{t\to\infty} f_t = M_F$ where f_t is the solution of (1.19) with $f_0 = F \in \mathscr{B}_2(\mathbf{R}^3)$. These rates both depend on the initial datum F, and on the collision kernel B. We obtain bounds on these rates that depend on the initial datum F only through the rate at which $\lim_{r\to 0} J_F(r) = 0$ where

$$J_F(r) = \int_{|v| > 1/r} |v|^2\, dF(v), \quad r > 0; \qquad J_F(0) = 0, \tag{1.20}$$

and only on B through the rate at which $\lim_{r\to 0} \Omega_B(r) = 0$ where

$$\Omega_B(r) = \sup_{|h| \leqslant r} \|B(\cdot + h) - B\|_{L^1(\mathbf{R})}, \qquad r \geqslant 0. \tag{1.21}$$

Notice that for all $F \in \mathscr{B}_2(\mathbf{R}^3)$, $J_F(r)$ is well defined for all $r > 0$, and it is always the case, by the dominated convergence theorem, that $\lim_{r\to 0} J_F(r) = 0$. However, the rate at which this convergence takes place can be arbitrarily slow in the class $\mathscr{B}_2(\mathbf{R}^3)$. A slow rate of convergence in $\lim_{r\to 0} J_F(r) = 0$ means that a large part of the energy distribution is concentrated in a small part of the mass distribution, out at high energies. Speaking more loosely, the relaxation to equilibrium is slow when a significant fraction of the energy is concentrated in a small fraction of the molecules.

Also note that only under the assumption that B is integrable, it will be the case that $\lim_{r\to 0} \Omega_B(r) = 0$. However, once again, the rate may be arbitrarily slow. It is convenient to define

$$\Omega_B^*(r) = \Omega_B(r)\,|\log \Omega_B(r)|. \tag{1.22}$$

We may now state our main result:

Theorem 1. Let $F \in P_2(\mathbf{R}^3; v_0, T)$ with $T > 0$, and let $M_F \in P_2(\mathbf{R}^3; v_0, T)$ be the corresponding Maxwellian distribution. Then

$$\|Q_n^+(F) - M_F\|_{\mathscr{B}_2} \leqslant \Phi_{B,F}(n^{-\alpha}), \qquad n = 1, 2, 3,... \tag{1.23}$$

where

$$\Phi_{B,F}(r) = C_{B,F}\Omega_B^*((r + J_F(r))^{1/60}), \qquad r \geqslant 0$$

$$C_{B,F} = C_B(1 + |v_0|)^3 (1+T)^2 \left(\int_{\mathbf{R}^3} |v - v_0| \, dF(v)\right)^{-1/30},$$

$\alpha = 60b/(120 + b)$, the constants $0 < b < 1$ and $0 < C_B < \infty$ depend only on the kernel $B(\cdot)$.

As a consequence of this we shall deduce:

Theorem 2. Let $F \in P_2(\mathbf{R}^3; v_0, T)$ with $T > 0$, and let f_t be the distributional solution of the Boltzmann equation (1.19) in $C([0, \infty); \mathscr{B}_2(\mathbf{R}^3, \|\cdot\|_{\mathscr{B}_2})$ with the initial datum $f_t|_{t=0} = F$. Let M_F be the corresponding Maxwellian distribution in $P_2(\mathbf{R}^3; v_0, T)$. Then

$$\|f_t - M_F\|_{\mathscr{B}_2} \leqslant \Phi_{B,F}(e^{-\beta t}), \qquad t \geqslant 0 \tag{1.24}$$

with $\beta = 30b/(60 + b)$, where the constant $b > 0$ and the function $\Phi_{B,F}(r)$ are given in Theorem 1 with the only difference that the constant C_B for defining $\Phi_{B,F}(r)$ is now replaced by $120 + C_B$.

Corollary to Theorem 2. Under the conditions in Theorem 2, suppose further that for some $\delta > 0$ the initial datum F satisfy

$$\int_{\mathbf{R}^3} |v|^{2+\delta} \, dF(v) < \infty$$

and the collision kernel $B(\cdot)$ satisfy the Hölder condition in L^1-norm: For some constants $0 < C_B < \infty$, $0 < \alpha = \alpha_B \leqslant 1$

$$\|B(\cdot + h) - B\|_{L^1(\mathbf{R})} \leqslant C_B \, |h|^\alpha, \qquad h \in \mathbf{R}. \tag{1.25}$$

Then

$$\|f_t - M_F\|_{\mathscr{B}_2} \leqslant Ce^{-\beta t}, \qquad t \geqslant 0$$

where the constant $\beta > 0$ depends only on δ and $B(\cdot)$, and the constant $C < \infty$ depends only on $B(\cdot)$ and F.

As an application of these explicit bounds on the convergence rate, we obtain the following estimate of global stability around any given solution whose initial datum is not a Dirac measure. (We explain the restriction in Section 6).

Theorem 3. If $0 \leqslant F \in \mathscr{B}_2(\mathbf{R}^3)$ is not a Dirac measure and $\|F\|_{\mathscr{B}_0} \neq 0$, then for any $0 \leqslant G \in \mathscr{B}_2(\mathbf{R}^3)$

$$\sup_{t \geqslant 0} \|f_t - g_t\|_{\mathscr{B}_2} \leqslant \Psi_{B,F}(\|F-G\|_{\mathscr{B}_2}) \tag{1.26}$$

where f_t, g_t are solutions of the Boltzmann equation (1.19) in $C([0,\infty); \mathscr{B}_2(\mathbf{R}^3, \|\cdot\|_{\mathscr{B}_2})$ with $f_t|_{t=0} = F$, $g_t|_{t=0} = G$, and

$$\Psi_{B,F}(r) = C_{B,F}\{r + \Omega_B^*((r^\alpha + J_F(r^\alpha))^{1/60})\}, \qquad r \geqslant 0$$

$$\alpha = \frac{\beta}{10\,\frac{\|F\|_{\mathscr{B}_2}}{\|F\|_{\mathscr{B}_0}} + \frac{\beta}{60}} > 0, \qquad C_{B,F} = C_B C_F,$$

$$C_F = C\left(|u_F|, T_F, \|F\|_{\mathscr{B}_0}, \int_{\mathbf{R}^3} |v - u_F|\, dF(v)\right) < \infty,$$

and u_F is the mean velocity, T_F is the temperature, and the constants $\beta > 0$, $C_B < \infty$ depend only on the kernel $B(\cdot)$; the function $(y_1, y_2, y_3, y_4) \mapsto C(y_1, y_2, y_3, y_4)$ can be written as an explicit function of (y_1, y_2, y_3, y_4) and is continuous on $[0,\infty) \times (0,\infty)^3$.

The fact that the stability holds in the $\|\cdot\|_{\mathscr{B}_2}$ norm, the energy norm, enables us to prove a bound on the "tails" of the energy distribution that hold globally in time. First recall[14] that for any $s > 2$, if $\int_{\mathbf{R}^3} |v|^s\, dF(v) < \infty$, the solution f_t with initial datum F has the property that $\sup_{t \geqslant 0} \int_{\mathbf{R}^3} |v|^s\, df_t(v) < C$ for some finite constant depending only on B and $\int_{\mathbf{R}^3} |v|^s\, dF(v)$. In the case of hard potentials, the evolution actually improves matters, producing moments of all orders at any finite positive time, assuming only that F has finite energy.[32] This is not true for Maxwellian molecules. For instance, if $\int_{\mathbf{R}^3} |v|^2 \log(1+|v|^2)\, dF(v) = \infty$, then $\int_{\mathbf{R}^3} |v|^2 \log(1+|v|^2)\, df_t(v) = \infty$ for all $t \geqslant 0$ because $df_t(v) \geqslant \exp(-\rho_F t)\, dF(v)$. Bobylev[7] describes this as an increase in the "tail temperature." The next result says that the tails cannot get too hot:

Corollary to Theorem 3. Let F be given in Theorem 3, let $f_t \in C([0, \infty); \mathscr{B}_2(\mathbf{R}^3, \|\cdot\|_{\mathscr{B}_2}))$ be the unique solution of Eq. (1.19) with $f_t|_{t=0} = F$. Then

$$\sup_{t \geqslant 0} \int_{|v| > R} |v|^2 \, df_t(v) \leqslant \Psi_{B,F}(2J_F(R^{-1/2})) + 2\, \|F\|_{\mathscr{B}_2} \frac{1}{R} \qquad \forall R > 0$$

where $\Psi_{B,F}(r)$ is the function given in Theorem 3.

We also obtain lower bounds that reveal the sharpness of the bounds stated above. We give a construction of initial data through which an arbitrarily slow rate of approach to equilibrium can be obtained. *In fact, this shows for the first time that there are solutions for which the rate of convergence in* $\lim_{t \to \infty} f_t = M_F$ *is worse than exponential.*

Bobylev[7] had earlier shown that for an arbitrary $\lambda > 0$, initial data F can be constructed with $\|f_t - M_F\|_{L^1} \geqslant Ce^{-\lambda t}$. However, until now, there was no proof that algebraic rates, or worse, could actually hold. Moreover, the lower bounds obtained here show that the upper bounds obtained in Theorem 2 are qualitatively best possible, at least when the kernel $B(\cdot)$ satisfies the Hölder condition (1.25). That is, at least in this case, the lower bounds and the upper bounds can be *scale-equivalent*. The meaning of this term is as follows:

Let $\Phi(t), \Psi(t)$ be nonnegative functions on $[0, \infty)$. Define a partial order $\preccurlyeq$ between Φ and Ψ by

$$\Phi(t) \preccurlyeq \Psi(t) \Leftrightarrow \Phi(t) \leqslant C[\Psi(\alpha t)]^{\beta} \;\; \forall t \in [0, \infty) \quad \text{for some } 0 < \alpha, \beta, \; C < \infty.$$

We say that Φ *and* Ψ *are scale-equivalent on* $[0, \infty)$ in case both $\Phi \preccurlyeq \Psi$ and $\Psi \preccurlyeq \Phi$ hold on $[0, \infty)$.

Recall that a function A on $[0, \infty)$ is *completely monotone* if it is a positive function in $C^{\infty}(0, \infty) \cap C[0, \infty)$ satisfying

$$(-1)^n \frac{d^n}{dt^n} A(t) \geqslant 0 \;\; \forall n = 0, 1, 2,...; \;\; \forall t > 0; \;\; A(0) = 1, \quad \text{and} \quad \lim_{t \to \infty} A(t) = 0.$$

A classical theorem of Bernstein[15] characterizes such functions as the Laplace transforms of finite positive measures on $[0, \infty)$. It follows that the decay rate of a completely monotone function is never faster than $e^{-\alpha t}$ for some constant $0 < \alpha < \infty$. Instead, it can be arbitrarily slow:

$$A(t) = (1+t)^{-\delta}, \quad (1+\log(1+t))^{-\delta}, \quad (1+\log(1+\log(1+t)))^{-\delta}, ... \qquad (1.27)$$

are all completely monotone. Indeed, Lemma 5.4 of Section 5 states the following: *If* $A_0(t)$ *is any completely monotone function, then so is* $A_1(t) = A_0(\log(1+t))$.

One more set of definitions is required for a complete statement of our lower bounds. Let

$$H^s(\mathbf{R}^3) = \left\{ f \mid (\|f\|_{H^s})^2 := \int_{\mathbf{R}^3} (1+|\xi|^2)^s \, |\hat{f}(\xi)|^2 \, d\xi < \infty \right\}, \quad s \geqslant 0$$

where $f \in L^1(\mathbf{R}^3)$ or $f \in \mathscr{B}_0(\mathbf{R}^3)$,

$$\hat{f}(\xi) = \int_{\mathbf{R}^3} e^{-\mathrm{i}\langle v, \xi\rangle} f(v) \, dv \qquad \text{resp.} \quad \hat{f}(\xi) = \int_{\mathbf{R}^3} e^{-\mathrm{i}\langle v, \xi\rangle} \, df(v)$$

denote the Fourier transforms of an integrable function f, or, respectively, bounded measure df. Supposing that $H \in \mathscr{B}_2(\mathbf{R}^3)$ and that $\int_{\mathbf{R}^3} (1, v) \, dH(v) = (0, 0)$, we can define $|||H|||_0$ by

$$|||H|||_0 := \sup_{0 \neq \xi \in \mathbf{R}^3} \frac{|\hat{H}(\xi)|}{|\xi|^2}. \tag{1.28}$$

This norm defines a weak metric appropriate to the study of $f_t - M_F$. It will also play an important role in the proof of Theorem 1. We may now state our lower bounds:

Theorem 4. Let $A(t)$ be any given completely monotone function on $[0, \infty)$. Then there is a strictly positive isotropic initial datum $F \in L^1_2(\mathbf{R}^3) \cap H^\infty(\mathbf{R}^3)$ and F is analytic on $\mathbf{R}^3$ satisfying $\int_{\mathbf{R}^3}(1, v, |v|^2) \, F(v) \, dv = (1, 0, 3)$, such that the unique solution $f_t(v)$ of the Boltzmann equation (1.1) in $C^1([0, \infty); L^1_2(\mathbf{R}^3))$ with initial condition $f_t|_{t=0} = F$ satisfies

$$\|f_t - M_F\|_{L^1} \geqslant \|\hat{f}_t - \widehat{M_F}\|_{L^\infty} \geqslant cA(t) \qquad \forall t \geqslant 0 \tag{1.29}$$

$$\|f_t - M_F\|_{L^1_2} \geqslant |||f_t - M_F|||_0 \geqslant cA(t) \qquad \forall t \geqslant 0 \tag{1.30}$$

for some constant $c > 0$. Furthermore, if the collision kernel $B(\cdot)$ satisfies the Hölder condition given in the Corollary to Theorem 2, then the upper bounds (1.24) and the lower bounds (1.29)–(1.30) are scale-equivalent. More precisely, the following functions

$$\|f_t - M_F\|_{L^1_2}, \ \|f_t - M_F\|_{L^1}, \ \Phi_{B,F}(e^{-t}), \ J_F(e^{-t}), \ A(t), \ \|\hat{f}_t - \widehat{M_F}\|_{L^\infty}, \ |||f_t - M_F|||_0$$

are pair wise scale-equivalent. Here the function $\Phi_{B,F}(\cdot)$ is given in Theorem 2.

Note that this theorem, together with the examples in (1.27), shows that *even for weak metric based on* $|||\cdot|||_0$, the convergence rate, whatever the smoothness of kernels, can be arbitrarily bad!

This is very different from what is known about solutions of the Boltzmann equation for hard spheres, for which $B(v-v_*, \sigma)$ is a multiple of $|v-v_*|$. In this case, it is known that for all initial data $0 \leqslant F \in L^1_2(\mathbf{R}^3)$, the convergence to equilibrium in the L^1-topology is always exponential (see ref. 1 and related results[5, 18, 30, 31]). Theorem 4 shows that matters are quite different in the Maxwellian case, and because of the relation between L^1-distance and the relative entropy, it shows that under just a finite energy condition, the entropy may relax to its equilibrium value arbitrarily slowly. While there are known examples to Cercignani's conjecture (refs. 8 and 35) concerning entropy production, these do not preclude that bad initial data eventually evolves into nice initial data with good entropy production. Theorem 4 does preclude this.

Theorem 4, together with Theorem 2 and its Corollary show that, for a fixed Maxwellian Kernel B, the key property of the initial datum F that determines the strong convergence rate is the decay rate of the function $J_F(r)$ as $r \to 0+$. This is sort of a generalized "moment condition," and it is what matters rather than any other properties such as an entropy condition, smoothness, etc. (In fact, the initial datum constructed to prove Theorem 4 is in $C^\infty \cap L^\infty$.) Evidently, an essential difference between "hard" potentials such as hard spheres, is that in the hard case, the function $J_{f_t}(r)$ *improves* as t increases, while it can deteriorate in the Maxwellian case, as discussed above the Corollary to Theorem 3.

It may be interesting to note that the method used for the Maxwellian molecules model does not rely on the strict positivity (almost everywhere) of the collision kernel $B(\cdot)$ (i.e., nonnegativity is enough). For the entropy method, which is so far still the only (direct or indirect) method that can deal with general kernels, certain "strong" positivity is needed in order to determine the entropy production. (See refs. 10, 11, 28, and 35).

1.4. Methods and Organization

Many features of Maxwellian molecules that we use here have already been introduced. Among those that have not is the *fundamental contraction property* of Maxwellian molecules, discovered by Tanaka.[25, 26] He proved that if f_t and g_t are two solutions of (1.19) with initial data F and G respectively, then for all $t > 0$

$$d_2(f_t, g_t) \leqslant d_2(F, G).$$

Here, d_2 is a metric on $\mathscr{B}_2(\mathbf{R}^3)$ often called the 2-Wasserstein distance. As shown in ref. 27, this metric is equivalent with that induced on $\mathscr{B}_2(\mathbf{R}^3)$ by the norm $|||\cdot|||_0$. Moreover, it had already been shown in ref. 16 that in this equivalent metric, the contraction property still holds.

The strategy behind the proof of Theorem 1 may be summarized as follows. First, in Section 2, we study the regularizing properties of repeated Wild convolution (see Lemma 2.1). We then use this regularization to prove that if $\lim_{n\to\infty} |||Q_n^+(F)-M_F|||_0 = 0$, then $\lim_{n\to\infty} \|Q_n^+(F)-M_F\|_{\mathscr{B}_0} = 0$. Moreover, the rates of convergence are related in a quantitative way depending only on the function Ω_B defined in (1.21). This is given by Lemma 2.5. In this part of the work, and elsewhere, we make extensive use of the Fourier transform of $Q^+(f, g)$, and Bobylev's formula[6] for this in the Maxwellian case, (2.1).

In Section 3 we prove bounds on the rate at which $\lim_{n\to\infty} |||Q_n^+(F)-M_F|||_{\mathscr{B}_0} = 0$. Here is where the contraction property plays a crucial role. We take general initial data $F \in P_2(\mathbf{R}^3; 0, 1)$, and approximate it by F_ε in the $|||\cdot|||_0$ norm where F_ε has finite fourth moments. The function J_F enters here, and determines the size of the fourth moment for a given degree of approximation. We can further arrange that $M_{F_\varepsilon} = M_F$. See Lemma 3.4 for the details. Then, further developing methods from refs. 16 and 12, it is possible to estimate the rate at which $|||Q_n^+(F_\varepsilon)-M_F|||_0$ tends to zero, using the fourth moment. Because of the contraction property, if both $|||F_\varepsilon - F|||_0$ and $|||Q_n^+(F_\varepsilon)-M_F|||_0$ are small, so is $|||Q_n^+(F)-M_F|||_0$. This is given in a series of lemmas, culminating in Lemma 3.3

Section 4 begins by putting together the ideas discussed above to prove Theorem 1. There is one more ingredient that deserves mention in the introduction. Notice that Theorem 1 estimates the rate of convergence in the $\|\cdot\|_{\mathscr{B}_2}$ norm, not the total variation norm $\|\cdot\|_{\mathscr{B}_0}$. However, Lemma 4.1 tells us that for all G in $P_2(\mathbf{R}^3; 0, 1)$

$$\|G-M\|_{\mathscr{B}_2} \leqslant C_0 \, \|G-M\|_{\mathscr{B}_0} \log\left(\frac{2e}{\|G-M\|_{\mathscr{B}_0}}\right)$$

where M is the Maxwellian in $P_2(\mathbf{R}^3; 0, 1)$. After taking into account the affine transformation needed to transform a general distribution in $\mathscr{B}_2(\mathbf{R}^3)$ into one in $P_2(\mathbf{R}^3; 0, 1)$, we obtain Theorem 1. It is through this lemma that Ω_B^* enters our estimates. Theorems 2 and its corollary are then deduced from Theorem 1 using the Wild's sum formula.

In Section 5 we prove Theorem 4. This consists of several steps. Among the most novel is a pointwise comparison property for solutions with the same initial data of the linearized Boltzmann equation and the full non-linear Boltzmann equation. The comparison is made in the Fourier representation; see Lemma 5.2. Lemma 5.4, assuring the complete monotonicity of $A_0(\log(1+t))$ when A_0 is completely monotone, is proved here as well.

Finally, in Section 6 we prove Theorem 3, the global stability in $\|\cdot\|_{\mathscr{B}_2}$, and its corollary. We emphasize that the corollary to Theorem 3 depends on the fact that we have global stability in the $\|\cdot\|_{\mathscr{B}_2}$ norm, and not just the total variation norm $\|\cdot\|_{\mathscr{B}_0}$. This is necessary to control the energy "tails." The strategy is similar to that which led to Theorem 1: Global control of the energy tails is easy and well known when the initial data has a finite fourth moment. Hence we approximate general finite energy initial data in the $\|\cdot\|_{\mathscr{B}_2}$ norm by initial data F_R with a finite fourth moment. Because of the global stability in $\|\cdot\|_{\mathscr{B}_2}$, the energy tails of the true solution cannot be much worse than those of the approximate solution.

2. ESTIMATE OF $\|Q_n^+(F) - M_F\|_{\mathscr{B}_0}$ BY $|||Q_n^+(F) - M_F|||_0$

We will make extensive use of the Fourier transform, and hence we recall the Bobylev identity:[6, 27] For any $B \in L^1[-1,1]$

$$Q^+(F,G)^\wedge(\xi) = \int_{\mathbf{S}^2} B(\langle \xi/|\xi|, \sigma\rangle)\, \hat{F}(\xi_+)\, \hat{G}(\xi_-)\, d\sigma, \quad F, G \in \mathscr{B}_0(\mathbf{R}^3) \tag{2.1}$$

where

$$\xi_+ = \frac{\xi + |\xi|\,\sigma}{2}, \qquad \xi_- = \frac{\xi - |\xi|\,\sigma}{2}.$$

In deriving the Bobylev identity (2.1), use is made of the following identity:

$$\int_{\mathbf{S}^2} B(\langle \omega, \sigma\rangle)\, \psi(\langle \zeta, \sigma\rangle)\, d\sigma = \int_{\mathbf{S}^2} B(\langle \zeta, \sigma\rangle)\, \psi(\langle \omega, \sigma\rangle)\, d\sigma \quad \forall \omega, \zeta \in \mathbf{S}^2 \tag{2.2}$$

where ψ is any bounded Borel function.

Our first Lemma is an estimate on the regularity produced by repeated Wild convolution when B is smooth, and small relative velocities are eliminated:

Lemma 2.1. Suppose the kernel B is smooth: $B \in C^1([-1,1])$. Let $0 < \delta \leqslant 1$ be a constant, and let $Q^+(\cdot,\cdot)$ and $Q^{\delta+}(\cdot,\cdot)$ be collision operators with the kernel $B(\langle z/|z|, \sigma\rangle)$ and $B(\langle z/|z|, \sigma\rangle)\, 1_{|z|\geqslant\delta}$ $(z = v - v_*)$ respectively. Then for any $0 < s < 1/2$, the 4-linear operator

$$(F_1, F_2, F_3, F_4) \mapsto Q^+(Q^{\delta+}(F_1, F_2), Q^{\delta+}(F_3, F_3))$$

is bounded from $[\mathscr{B}_0(\mathbf{R}^3)]^4$ into $H^s(\mathbf{R}^3)$:

$$\|Q^+(Q^{\delta+}(F_1, F_2), Q^{\delta+}(F_3, F_4))\|_{H^s} \leqslant C_{B,s}\, \delta^{-2} \prod_{i=1}^{4} \|F_i\|_{\mathscr{B}_0}$$

where

$$C_{B,s} = \frac{C_0}{\sqrt{1-2s}} (\|B\|_{1,\infty})^2 \, \|B\|_{L^1([-1,1];(1-t^2)^{-1/2}\,dt)},$$

C_0 is an absolute constant and $\|B\|_{1,\infty} = \|B\|_{L^\infty[-1,1]} + \|\frac{d}{dt} B\|_{L^\infty[-1,1]}$.

Proof. We first estimate $Q^{\delta+}(F, G)^\wedge(\xi)$ for $F, G \in \mathscr{B}_0(\mathbf{R}^3)$. In the formula (1.17), take $\phi(v) = e^{-\mathrm{i}\langle \xi, v\rangle}$ and let z denote $v - v_*$. We have

$$Q^{\delta+}(F, G)^\wedge(\xi)$$
$$= \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \left[\int_{\mathbf{S}^2} B(\langle z/|z|, \sigma\rangle)\, e^{-\frac{\mathrm{i}}{2}|z|\langle \xi, \sigma\rangle}\, d\sigma \right] 1_{\{|z| \geqslant \delta\}} e^{-\frac{\mathrm{i}}{2}\langle \xi, v+v_*\rangle}\, dF(v)\, dG(v_*).$$

To compute the inner integral, parameterize $\mathbf{S}^2$ as follows: Let

$$\sigma = \cos(\theta) \frac{\xi}{|\xi|} + \sin(\theta)[\cos(\phi)\, \mathbf{i} + \sin(\phi)\, \mathbf{j}], \qquad \theta \in [0, \pi], \quad \phi \in [0, 2\pi]$$

where $\{\xi/|\xi|, \mathbf{i}, \mathbf{j}\}$ is an orthonormal base of $\mathbf{R}^3$, then $\langle \xi, \sigma\rangle = |\xi| \cos(\theta)$ and

$$\langle z/|z|, \sigma\rangle = \langle z/|z|, \xi/|\xi|\rangle \cos(\theta) + \sqrt{1 - \langle z/|z|, \xi/|\xi|\rangle^2} \sin(\theta) \cos(\phi - \alpha)$$

here α is independent of ϕ. This gives

$$\int_{\mathbf{S}^2} B(\langle z/|z|, \sigma\rangle)\, e^{-\frac{\mathrm{i}}{2}|z|\langle \xi, \sigma\rangle} d\sigma = \int_0^\pi \sin(\theta)\, e^{-\frac{\mathrm{i}}{2}|z|\,|\xi| \cos(\theta)} A_{z,\xi}(\theta)\, d\theta$$

where

$$A_{z,\xi}(\theta) = \int_0^{2\pi} B(\langle z/|z|, \xi/|\xi|\rangle \cos(\theta) + \sqrt{1 - \langle z/|z|, \xi/|\xi|\rangle^2} \sin(\theta) \cos(\phi))\, d\phi.$$

Integration by parts gives

$$\int_0^\pi \sin(\theta)\, e^{-\frac{\mathrm{i}}{2}|z|\,|\xi| \cos(\theta)} A_{z,\xi}(\theta)\, d\theta$$
$$= \frac{2}{\mathrm{i}\,|z|\,|\xi|} e^{-\frac{\mathrm{i}}{2}|z|\,|\xi|\cos(\theta)} A_{z,\xi}(\theta) \Big|_{\theta=0}^{\theta=\pi} - \frac{2}{\mathrm{i}\,|z|\,|\xi|} \int_0^\pi e^{-\frac{\mathrm{i}}{2}|z|\,|\xi|\cos(\theta)} \frac{d}{d\theta} A_{z,\xi}(\theta)\, d\theta.$$

By the inequalities $|A_{z,\xi}(\theta)| \leqslant 2\pi \, \|B\|_{L^\infty[-1,1]}$ and $|\frac{d}{d\theta} A_{z,\xi}(\theta)| \leqslant 2\pi \, \|\frac{d}{dt} B\|_{L^\infty[-1,1]}$, we obtain

$$\left| \int_{S^2} B(\langle z/|z|, \sigma\rangle) \, e^{-\frac{i}{2}|z| \langle \xi, \sigma\rangle} \, d\sigma \right| \leqslant \frac{4\pi^2 \, \|B\|_{1,\infty}}{|z| \, |\xi|}.$$

This is a good bound for large $|\xi|$. For small $|\xi|$, the left hand side is trivially bounded by $4\pi \, \|B\|_{L^\infty[-1,1]}$. Together, these estimates yield

$$\left| \int_{S^2} B(\langle z/|z|, \sigma\rangle) \, e^{-\frac{i}{2}|z| \langle \xi, \sigma\rangle} \, d\sigma \right| \leqslant 8\pi^2 \, \|B\|_{1,\infty} \, \delta^{-1} \frac{1}{1+|\xi|}, \qquad |z| \geqslant \delta.$$

Therefore,

$$|Q^{\delta+}(F, G)^\wedge(\xi)| \leqslant 8\pi^2 \, \|B\|_{1,\infty} \, \delta^{-1} \, \|F\|_{\mathscr{B}_0} \, \|G\|_{\mathscr{B}_0} \frac{1}{1+|\xi|}, \qquad \xi \in \mathbf{R}^3. \quad (2.3)$$

Now applying this estimate at ξ_+ and ξ_- we get

$$\begin{aligned} &|Q^{\delta+}(F_1, F_2)^\wedge(\xi_+)| \, |Q^{\delta+}(F_3, F_4)^\wedge(\xi_-)| \\ &\quad \leqslant 64\pi^4 (\|B\|_{1,\infty})^2 \, \delta^{-2} \left(\prod_{i=1}^{4} \|F_i\|_{\mathscr{B}_0} \right) \frac{1}{(1+|\xi_+|)(1+|\xi_-|)}. \end{aligned}$$

Since $(1+|\xi_+|)(1+|\xi_-|) \geqslant 1 + \frac{1}{2} |\xi|^2 \sin(\theta)$, it follows from the Bobylev identity (2.1) that

$$\begin{aligned} &|Q^+(Q^{\delta+}(F_1, F_2), Q^{\delta+}(F_3, F_4))^\wedge(\xi)| \\ &\quad \leqslant 64\pi^4 (\|B\|_{1,\infty})^2 \, \delta^{-2} \left(\prod_{i=1}^{4} \|F_i\|_{\mathscr{B}_0} \right) 2\pi \int_0^\pi B(\cos(\theta)) \frac{\sin(\theta) \, d\theta}{1 + \frac{1}{2}|\xi|^2 \sin(\theta)} \\ &\quad \leqslant 2^8 \pi^5 (\|B\|_{1,\infty})^2 \left(\int_0^\pi B(\cos(\theta)) \, d\theta \right) \delta^{-2} \left(\prod_{i=1}^{4} \|F_i\|_{\mathscr{B}_0} \right) \frac{1}{1+|\xi|^2}. \end{aligned} \quad (2.4)$$

This gives the H^s-bounds since for $0 < s < 1/2$, $\int_{\mathbf{R}^3} (1+|\xi|^2)^{s-2} \, d\xi \leqslant 8\pi/(1-2s)$. ∎

Remark. In fact, the proof gives pointwise estimates (2.3)–(2.4) for the Fourier transforms of the convolved measures. As one sees, the single convolution estimate (2.3) does not give a sufficiently strong pointwise estimate to obtain the regularity that we require here, though under other conditions, results have been obtained in this case; see refs. 20, 31, 9, and 21.

Before stating our next lemma, we recall some properties of convolution of measures with functions: Let $\mu \in \mathscr{B}_s(\mathbf{R}^3)$ and $\phi \in L^1_s(\mathbf{R}^3)$ $(s \geqslant 0)$. The convolution $\mu * \phi$ is a Borel measure defined by (due to Riesz representation theorem)

$$\int_{\mathbf{R}^3} \psi(v)\, d(\mu * \phi)(v) = \int_{\mathbf{R}^3} \left(\int_{\mathbf{R}^3} \psi(v+v_*)\, \phi(v_*)\, dv_* \right) d\mu(v)$$

for all Borel functions ψ satisfying $\sup_{v \in \mathbf{R}^3} |\psi(v)|\, (1+|v|^2)^{-s/2} < \infty$. From this definition we have $|\mu * \phi| \leqslant |\mu| * |\phi|$, $\mu * \phi \in \mathscr{B}_s(\mathbf{R}^3)$ and $\|\mu * \phi\|_{\mathscr{B}_s} \leqslant \|\mu\|_{\mathscr{B}_s} \|\phi\|_{L^1_s}$. And if $\mu \geqslant 0$, $\phi \geqslant 0$, then $\mu * \phi \geqslant 0$ and $\|\mu * \phi\|_{\mathscr{B}_0} = \|\mu\|_{\mathscr{B}_0} \|\phi\|_{L^1}$. Moreover if $\phi, \psi \in C_c^\infty(\mathbf{R}^3)$ then

$$\widehat{F * \phi}\,(\xi) = \hat{F}(\xi)\, \hat{\phi}(\xi), \qquad \langle F, \psi \rangle_0 = (2\pi)^{-3} \int_{\mathbf{R}^3} \hat{F}(\xi)\, \hat{\psi}(-\xi)\, d\xi.$$

Let $0 \leqslant \phi_1 \in C_c^\infty(\mathbf{R}^3)$ be defined by $\phi_1(z) = c \exp\{-1/(1-|z|^2)\}$ for $|z| < 1$, $\phi_1(z) = 0$ for $|z| \geqslant 1$, and $\int_{\mathbf{R}^3} \phi_1(z) = 1$. For any $\varepsilon > 0$, let $\phi_\varepsilon(z) = \varepsilon^{-3} \phi_1(z/\varepsilon)$.

Lemma 2.2.

(i) Let $F \in P_2(\mathbf{R}^3; 0, 1)$. Then

$$\sup_{v \in \mathbf{R}^3} \int_{|v_* - v| < \delta} dF(v_*) \leqslant C_0(\delta^{-2} |\!|\!| F - M_F |\!|\!|_0 + \delta^3) \qquad \forall \delta > 0, \tag{2.5}$$

$$\|(F - M_F) * \phi_\varepsilon\|_{\mathscr{B}_0} \leqslant C_0 \varepsilon^{-2} (|\!|\!| F - M_F |\!|\!|_0)^{4/7} \qquad \forall 0 < \varepsilon \leqslant 2. \tag{2.6}$$

(ii) Let $F \in \mathscr{B}_2(\mathbf{R}^3) \cap H^s(\mathbf{R}^3)$, $0 < s \leqslant 2$. Then

$$\|F * \phi_\varepsilon - F\|_{\mathscr{B}_0} \leqslant C_0 (\|F\|_{\mathscr{B}_2})^{3/7} (\|F\|_{H^s})^{4/7}\, \varepsilon^{4s/7} \qquad \forall 0 < \varepsilon \leqslant 2. \tag{2.7}$$

Here $C_0 > 0$ is an absolute constant, $\phi_\varepsilon \in C_c^\infty(\mathbf{R}^3)$ is given above.

Proof. (i) Consider the convolution $\psi_\delta = 1_{[0, 2\delta]}(|\cdot|) * \phi_\delta$, where $\phi_\delta(z) = \delta^{-3} \phi_1(z/\delta)$. We have $0 \leqslant \psi_\delta \in C_c^\infty(\mathbf{R}^3)$, $\psi_\delta(v) = 1$ for $|v| < \delta$, and

$$\widehat{\psi_\delta}\,(\xi) = (1_{[0, 2\delta]}(|\cdot|))^\wedge\,(\xi)\, \widehat{\phi_\delta}\,(\xi) = \delta^3 (1_{[0, 2]}(|\cdot|))^\wedge\,(\delta\xi)\, \widehat{\phi_1}\,(\delta\xi).$$

In the following, we denote by the same letter C_0 various absolute constants, and M_F will be denoted simply by M. Then by $F \geqslant 0$ and $1_{|v-v_*|<\delta} \leqslant \psi_\delta(v-v_*)$ and $\psi_\delta(v-\cdot) \in C_c^\infty(\mathbf{R}^3)$ we have

$$\int_{|v_*-v|<\delta} dF(v_*) \leqslant \int_{\mathbf{R}^3} \psi_\delta(v-v_*)\, dF(v_*) = \int_{\mathbf{R}^3} \psi_\delta(v-v_*)\, dF(v_*) - \int_{\mathbf{R}^3} \psi_\delta(v-v_*)\, M(v_*)\, dv_* + \int_{\mathbf{R}^3} \psi_\delta(v-v_*)\, M(v_*)\, dv_*,$$

and

$$\begin{aligned}
&\int_{\mathbf{R}^3} \psi_\delta(v-v_*)\, dF(v_*) - \int_{\mathbf{R}^3} \psi_\delta(v-v_*)\, M(v_*)\, dv_* \\
&= (2\pi)^{-3} \int_{\mathbf{R}^3} (\hat{F}(\xi) - \hat{M}(\xi))\, \widehat{\psi_\delta}(\xi)\, e^{i\langle \xi, v\rangle}\, d\xi \\
&\leqslant (2\pi)^{-3} \int_{\mathbf{R}^3} |\hat{F}(\xi) - \hat{M}(\xi)|\, |\widehat{\psi_\delta}(\xi)|\, d\xi \\
&= (2\pi)^{-3} \int_{\mathbf{R}^3} |\hat{F}(\xi) - \hat{M}(\xi)|\, \delta^3 (1_{[0,2]}(|\cdot|))^\wedge(\delta\xi)\,|\,|\widehat{\phi_1}(\delta\xi)|\, d\xi \\
&\leqslant (2\pi)^{-3}\, |||F-M|||_0\, \delta^3 \frac{4\pi}{3} 2^3 \int_{\mathbf{R}^3} |\xi|^2\, |\hat{\phi}_1(\delta\xi)|\, d\xi = C_0 \delta^{-2}\, |||F-M|||_0.
\end{aligned} \tag{2.8}$$

On the other hand,

$$\int_{\mathbf{R}^3} \psi_\delta(v-v_*)\, M(v_*)\, dv_* \leqslant \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbf{R}^3} \psi_\delta(v_*)\, dv_* = \frac{16}{3\sqrt{2\pi}} \delta^3.$$

This together with (2.8) gives (2.5).

Next, we prove (2.6). Using (1.13) and (1.15),

$$\|(F-M) * \phi_\varepsilon\|_{\mathscr{B}_0} = \sup_{\psi \in C_c^\infty(\mathbf{R}^3),\, \|\psi\|_{L^\infty} \leqslant 1} |\langle (F-M) * \phi_\varepsilon, \psi\rangle_0|.$$

For any $\psi \in C_c^\infty(\mathbf{R}^3)$ satisfying $\|\psi\|_{L^\infty} \leqslant 1$ and for any $R>0$, $\eta>0$, consider the decomposition:

$$\psi = \psi_{R,\eta}^{(1)} + \psi_{R,\eta}^{(2)}, \qquad \psi_{R,\eta}^{(1)} = \psi \chi_{R,\eta}, \qquad \psi_{R,\eta}^{(2)} = \psi(1-\chi_{R,\eta})$$

where $\chi_{R,\eta} \in C_c^\infty(\mathbf{R}^3)$ satisfies $0 \leqslant \chi_{R,\eta} \leqslant 1$ on $\mathbf{R}^3$, $\chi_{R,\eta}(v) = 1$ for $|v| \leqslant R$, and $\chi_{R,\eta}(v) = 0$ for $|v| \geqslant R + 2\eta$. Then $\psi_{R,\eta}^{(1)}, \psi_{R,\eta}^{(2)} \in C_c^\infty(\mathbf{R}^3)$ and $|\psi_{R,\eta}^{(1)}(v)| \leqslant 1_{|v| \leqslant R+2\eta}$, $|\psi_{R,\eta}^{(2)}(v)| \leqslant 1_{|v| \geqslant R}$. We have

$$|\langle (F-M) * \phi_\varepsilon, \psi \rangle_0| \leqslant |\langle (F-M) * \phi_\varepsilon, \psi_{R,\eta}^{(1)} \rangle_0| + |\langle (F-M) * \phi_\varepsilon, \psi_{R,\eta}^{(2)} \rangle_0|.$$

Using the Cauchy–Schwarz inequality, the Plancherel identity and $\widehat{\phi_\varepsilon}(\xi) = \widehat{\phi_1}(\varepsilon\xi)$, we have

$$\begin{aligned}
&|\langle (F-M) * \phi_\varepsilon, \psi_{R,\eta}^{(1)} \rangle_0|^2 \\
&\quad = (2\pi)^{-6} \left| \int_{\mathbf{R}^3} \widehat{\psi_{R,\eta}^{(1)}}(-\xi)\, \widehat{\phi_\varepsilon}(\xi) [\hat{F}(\xi) - \hat{M}(\xi)]\, d\xi \right|^2 \\
&\quad \leqslant (2\pi)^{-6} \left(\int_{\mathbf{R}^3} |\widehat{\psi_{R,\eta}^{(1)}}(-\xi)|^2\, d\xi \right) \left(\int_{\mathbf{R}^3} |\widehat{\phi_\varepsilon}(\xi) [\hat{F}(\xi) - \hat{M}(\xi)]|^2\, d\xi \right) \\
&\quad = (2\pi)^{-3} \left(\int_{\mathbf{R}^3} |\psi_{R,\eta}^{(1)}(v)|^2\, dv \right) \left(\int_{\mathbf{R}^3} |\widehat{\phi_1}(\varepsilon\xi) [\hat{F}(\xi) - \hat{M}(\xi)]|^2\, d\xi \right) \\
&\quad \leqslant (2\pi)^{-3} \left(\int_{\mathbf{R}^3} 1_{|v| \leqslant R+2\eta}\, dv \right) \left(\int_{\mathbf{R}^3} |\widehat{\phi_1}(\varepsilon\xi)|^2\, |\xi|^4\, \|\!|F-M|\!\|_0^2\, d\xi \right) \\
&\quad \leqslant (2\pi)^{-3} \frac{4\pi}{3} (R+2\eta)^3\, \|\!|F-M|\!\|_0^2\, \varepsilon^{-7} \int_{\mathbf{R}^3} |\xi|^4\, |\widehat{\phi_1}(\xi)|^2\, d\xi.
\end{aligned}$$

This gives $|\langle (F-M) * \phi_\varepsilon, \psi_{R,\eta}^{(1)} \rangle_0| \leqslant C_0 (R+2\eta)^{3/2} \varepsilon^{-7/2} \|\!|F-M|\!\|_0$. To estimate the second term, we use the inequality $\int_{\mathbf{R}^3} |v|^2\, d\, |G * \phi_\varepsilon|(v) \leqslant 2\, \|\phi_\varepsilon\|_{L_2^1}\, \|G\|_{\mathscr{B}_2}$, valid for all $G \in \mathscr{B}_2(\mathbf{R}^3)$. We then have (because $0 < \varepsilon \leqslant 2$)

$$\begin{aligned}
|\langle (F-M) * \phi_\varepsilon, \psi_{R,\eta}^{(2)} \rangle_0| &\leqslant \int_{\mathbf{R}^3} 1_{|v| \geqslant R}\, d\, |(F-M) * \phi_\varepsilon|(v) \\
&\leqslant R^{-2} \int_{\mathbf{R}^3} |v|^2\, d\, |(F-M) * \phi_\varepsilon|(v) \leqslant C_0 R^{-2}.
\end{aligned}$$

Thus,

$$|\langle (F-M) * \phi_\varepsilon, \psi \rangle_0| \leqslant C_0 (R+2\eta)^{3/2} \varepsilon^{-7/2} \|\!|F-M|\!\|_0 + C_0 R^{-2}$$

for all $R > 0, \eta > 0$. Letting $\eta \to 0$ and taking *sup* with respect to $\psi \in C_c^\infty(\mathbf{R}^3)$ satisfying $\|\psi\|_{L^\infty} \leqslant 1$ gives $\|(F-M) * \phi_\varepsilon\|_{\mathscr{B}_0} \leqslant C_0 (R^{3/2} \varepsilon^{-7/2} \|\!|F-M|\!\|_0 + R^{-2})$. Minimizing the last term with respect to $R > 0$ gives the inequality (2.6).

(ii) Suppose $F \in \mathscr{B}_2(\mathbf{R}^3) \cap H^s(\mathbf{R}^3)$, $0 < s \leqslant 2$. For any $\psi \in C_c^\infty(\mathbf{R}^3)$ satisfying $\|\psi\|_{L^\infty} \leqslant 1$, let $\psi_{R,\eta}^{(1)}$, $\psi_{R,\eta}^{(2)}$ be given above. Then,

$$|\langle F * \phi_\varepsilon - F, \psi \rangle_0| \leqslant |\langle F * \phi_\varepsilon - F, \psi_{R,\eta}^{(1)} \rangle_0| + |\langle F * \phi_\varepsilon - F, \psi_{R,\eta}^{(2)} \rangle_0|.$$

Estimating the first term on the right side:

$$\begin{aligned}
&|\langle F * \phi_\varepsilon - F, \psi_{R,\eta}^{(1)} \rangle_0|^2 \\
&\quad = (2\pi)^{-6} \left| \int_{\mathbf{R}^3} \hat{\psi}_{R,\eta}^{(1)}(-\xi)[\widehat{\phi_\varepsilon}(\xi) - 1]\, \hat{F}(\xi)]\, d\xi \right|^2 \\
&\quad \leqslant (2\pi)^{-3} \left(\int_{\mathbf{R}^3} |\psi_{R,\eta}^{(1)}(v)|^2\, dv \right) \left(\int_{\mathbf{R}^3} |\widehat{\phi_1}(\varepsilon\xi) - 1|^2\, |\hat{F}(\xi)|^2\, d\xi \right) \\
&\quad \leqslant (2\pi)^{-3} \left(\int_{\mathbf{R}^3} 1_{|v| \leqslant R+2\eta}\, dv \right) \left(\int_{\mathbf{R}^3} |\widehat{\phi_1}(\varepsilon\xi) - 1|^2\, |\hat{F}(\xi)|^2\, d\xi \right).
\end{aligned}$$

Since $\widehat{\phi_1}(0) = 1$, $\frac{\partial}{\partial \xi_j} \widehat{\phi_1}(0) = 0$ and the support of ϕ_1 is the unit ball, an estimate using Taylor's theorem provides $|\widehat{\phi_1}(\varepsilon\xi) - 1| \leqslant \min\{2, \frac{1}{2}\varepsilon^2 |\xi|^2\}$. The minimum of two positive numbers is less than any of their geometric means, so

$$\min\{2, h\} \leqslant 2^{1-\alpha} h^\alpha, \qquad h \geqslant 0, \quad 0 \leqslant \alpha \leqslant 1. \tag{2.9}$$

Choosing $h = \frac{1}{2}\varepsilon^2 |\xi|^2$, $\alpha = s/2$ we get $|\widehat{\phi_1}(\varepsilon\xi) - 1| \leqslant 2^{1-s}\varepsilon^s |\xi|^s$. This gives

$$\int_{\mathbf{R}^3} |\widehat{\phi_1}(\varepsilon\xi) - 1|^2\, |\hat{F}(\xi)|^2\, d\xi \leqslant 4 \left(\int_{\mathbf{R}^3} |\xi|^{2s}\, |\hat{F}(\xi)|^2\, d\xi \right) \varepsilon^{2s},$$

and thus $|\langle F * \phi_\varepsilon - F, \psi_{R,\eta}^{(1)} \rangle_0| \leqslant C_0 (R + 2\eta)^{3/2} \|F\|_{H^s} \varepsilon^s$.

The estimate for the second term is much the same: $|\langle F * \phi_\varepsilon - F, \psi_{R,\eta}^{(2)} \rangle_0| \leqslant \int_{|v| \geqslant R} d\,|F * \phi_\varepsilon - F| \leqslant C_0 \|F\|_{\mathscr{B}_2} R^{-2}$, so that $|\langle F * \phi_\varepsilon - F, \psi \rangle_0| \leqslant C_0 (R + 2\eta)^{3/2} \|F\|_{H^s} \varepsilon^s + C_0 \|F\|_{\mathscr{B}_2} R^{-2}$ for all $R > 0, \eta > 0$ and all $\psi \in C_c^\infty(\mathbf{R}^3)$ satisfying $\|\psi\|_{L^\infty} \leqslant 1$. Letting $\eta \to 0$ gives $\|F * \phi_\varepsilon - F\|_{\mathscr{B}_0} \leqslant C_0 R^{3/2} \|F\|_{H^s} \varepsilon^s + C_0 \|F\|_{\mathscr{B}_2} R^{-2}$ for all $R > 0$. Taking minimum for the right hand side with respect to $R > 0$, we obtain (2.7) . ▮

In proving our next results, we will require certain properties of the functions $\Omega_B(r)$ and $\Omega_B^*(r)$ defined in (1.21)–(1.22).

Lemma 2.3. The functions $\Omega_B(r), \Omega_B^*(r)$ are continuous, non-decreasing on $[0, \infty)$ and have the following properties:

$$\frac{r}{\pi(2+r)} \leqslant \Omega_B(r) \leqslant 1/\pi \quad \text{and} \quad \Omega_B(r) \leqslant \Omega_B^*(r), \qquad r \geqslant 0;$$

$$\Omega_B(\lambda r) \leqslant (1+\lambda)\,\Omega_B(r), \qquad \Omega_B^*(\lambda r) \leqslant (1+\lambda)\,\Omega_B^*(r), \qquad \lambda, r \geqslant 0.$$

Proof. By supp $B \subset [-1, 1]$ we have $\|B(\cdot+h)-B\|_{L^1(\mathbf{R})} \leqslant 2\int_{-1}^{1} B(\tau)\,d\tau = 1/\pi\ \forall h \in \mathbf{R}$ and $\Omega_B(2) \geqslant \|B(\cdot+2)-B\|_{L^1(\mathbf{R}^3)} = 2\int_{-1}^{1} B(\tau)\,d\tau = 1/\pi$. These imply that $0 \leqslant \Omega_B(r) \leqslant 1/\pi = \Omega_B(2)\ \forall r \geqslant 0$. Now the inequality $0 \leqslant \Omega_B(r) \leqslant 1/\pi$ implies that $|\log \Omega_B(r)| \geqslant 1$ and so $\Omega_B(r) \leqslant \Omega_B(r)\,|\log \Omega_B(r)| = \Omega_B^*(r)$ for all $r \geqslant 0$.

To prove the other properties, we use the subadditivity of Ω_B and Ω_B^*:

$$\Omega_B(r_1+r_2) \leqslant \Omega_B(r_1)+\Omega_B(r_2), \quad \Omega_B^*(r_1+r_2) \leqslant \Omega_B^*(r_1)+\Omega_B^*(r_2) \quad \forall r_1, r_2 \geqslant 0. \tag{2.10}$$

To see this, note that the function $y \mapsto y\,|\log y|$ is concave and increasing from zero on $[0, 1/e]$ which implies that $(y_1+y_2)\,|\log(y_1+y_2)| \leqslant y_1\,|\log y_1| + y_2\,|\log y_2|$, $y_1, y_2 \in [0, 1/e]$. Since $0 \leqslant \Omega_B(r) \leqslant 1/\pi < 1/e$ and the function $r \mapsto \Omega_B(r)$ is non-decreasing on $[0, \infty)$, it follows that the function $r \mapsto \Omega_B^*(r)$ is also non-decreasing on $[0, \infty)$ and the first inequality in (2.10) implies the second one. Thus we need only to prove the first inequality. Let $r_1, r_2 \geqslant 0$. We can assume that $r_1+r_2 > 0$. For any $h \in \mathbf{R}$ satisfying $|h| \leqslant r_1+r_2$, let $h_1 = \frac{r_1}{r_1+r_2} h$, $h_2 = \frac{r_2}{r_1+r_2} h$. Then $h = h_1+h_2$ and $|h_1| \leqslant r_1$, $|h_2| \leqslant r_2$. By definition of $\Omega_B(\cdot)$, this gives the first inequality in (2.10).

The inequalities in (2.10) and the monotonicity imply the continuity of $\Omega_B(r)$ and $\Omega_B^*(r)$. In fact, for instance for $\Omega_B(r)$ we have $|\Omega_B(r_1)-\Omega_B(r_2)| \leqslant \Omega_B(|r_1-r_2|)$, $r_1, r_2 \geqslant 0$.

From (2.10) we have $\Omega_B(nr) \leqslant n\Omega_B(r)$ for all $r \geqslant 0$, $n = 1, 2,...$ Let $\lambda, r \geqslant 0$ and let $[\lambda]$ be the largest integer not exceeding λ. Then, by monotonicity,

$$\Omega_B(\lambda r) \leqslant \Omega_B((1+[\lambda])\,r) \leqslant (1+[\lambda])\,\Omega_B(r) \leqslant (1+\lambda)\,\Omega_B(r).$$

Similarly we have $\Omega_B^*(\lambda r) \leqslant (1+\lambda)\,\Omega_B^*(r)$ for all $\lambda \geqslant 0, r \geqslant 0$. Using the this inequality, we have for all $r > 0$

$$1/\pi = \Omega_B(2) = \Omega_B\left(\frac{2}{r}\cdot r\right) \leqslant \left(1+\frac{2}{r}\right)\Omega_B(r) = \frac{r+2}{r}\,\Omega_B(r).$$

Since $\Omega_B(0) = 0$, this gives the inequality $r \leqslant \pi(r+2)\,\Omega_B(r)$ for all $r \geqslant 0$. ∎

For the later use, we observe the following:

$$F \in P_2(\mathbf{R}^3; 0, 1) \Rightarrow |||F - M_F|||_0 \leqslant \min\{2, \tfrac{1}{2}\, \|F - M_F\|_{\mathcal{B}_2}\}. \tag{2.11}$$

Indeed, by Taylor's theorem, we have that for $F \in P_2(\mathbf{R}^3; 0, 1)$,

$$\hat{F}(\xi) = 1 - \int_0^1 (1-\tau) \int_{\mathbf{R}^3} \langle v, \xi\rangle^2 e^{-i\langle \tau\xi, v\rangle}\, dF(v)\, d\tau. \tag{2.12}$$

Since M_F is isotropic, we have $\int_{\mathbf{R}^3} \langle v, \xi\rangle^2 M_F(v)\, dv = |\xi|^2$, $\xi \in \mathbf{R}^3$, and thus

$$\frac{|\hat{F}(\xi) - \widehat{M_F}(\xi)|}{|\xi|^2} \leqslant \frac{1}{2} \int_{\mathbf{R}^3} \langle v, \xi/|\xi|\rangle^2\, dF(v) + \frac{1}{2} \int_{\mathbf{R}^3} \langle v, \xi/|\xi|\rangle^2 M_F(v)\, dv \leqslant \frac{3}{2} + \frac{1}{2} = 2.$$

Hence $|||F - M_F|||_0 \leqslant 2$. Also, for all $\xi \in \mathbf{R}^3 \setminus \{0\}$

$$\frac{|\hat{F}(\xi) - \widehat{M_F}(\xi)|}{|\xi|^2} \leqslant \frac{1}{2} \int_{\mathbf{R}^3} \langle v, \xi/|\xi|\rangle^2\, d\,|F - M_F|\,(v)\, d\tau \leqslant \frac{1}{2}\, \|F - M_F\|_{\mathcal{B}_2}.$$

So $|||F - M_F|||_0 \leqslant \frac{1}{2}\, \|F - M_F\|_{\mathcal{B}_2}$.

Lemma 2.4. Let $F_i \in P_2(\mathbf{R}^3; 0, 1)$, $i = 1, 2, 3, 4$, and let $0 < q \leqslant 2$ satisfy

$$\min\{|||F_1 - M|||_0, |||F_2 - M|||_0\} \leqslant q \quad \text{and} \quad \min\{|||F_3 - M|||_0, |||F_4 - M|||_0\} \leqslant q$$

where M is the Maxwellian in $P_2(\mathbf{R}^3; 0, 1)$. Then for $\varepsilon = q^{2/7 - 1/120}$,

$$\|[(F_1 \circ F_2) \circ (F_3 \circ F_4)] * \phi_\varepsilon - (F_1 \circ F_2) \circ (F_3 \circ F_4)\|_{\mathcal{B}_0} \leqslant C_0 \Omega_B(q^{1/60})$$

where $\Omega_B(\cdot)$ is defined in (1.21), ϕ_ε is given above Lemma 2.2, and $C_0 < \infty$ is an absolute constant.

Note. The present lemma will be applied together with Lemma 2.5 below. The exponents such as $2/7 - 1/120$ and $1/60$ in these two lemmas are chosen to balance the effectiveness of these two lemmas together.

Proof. In the following, C_0 $(0 < C_0 < \infty)$ denotes an absolute constant that varies from line to line.

The first step is to approximate the given kernel B by one that is smmoth so that Lemma 2.1 may be applied. Let $\varphi_1(t) = c \exp\{-1/(1-t^2)\}$ for $|t| < 1$, $\varphi_1(t) = 0$ for $|t| \geqslant 1$, and choose c so that $\int_{\mathbf{R}} \varphi_1(t)\, dt = 1$. Let

$\varphi_\lambda(t) = \frac{1}{\lambda}\varphi_1(\frac{t}{\lambda})$, and $B_\lambda(t) = (B * \phi_\lambda)(t)$, $0 < \lambda \leqslant 2$. Then B_λ is also even and satisfies $\|B_\lambda\|_{L^1} = \|B\|_{L^1} = 1/2\pi$ and

$$\|B_\lambda - B\|_{L^1} \leqslant \Omega_B(\lambda), \qquad \|B_\lambda\|_{L^\infty} \leqslant C_0\lambda^{-1}, \qquad \|B_\lambda\|_{1,\infty} \leqslant C_0\lambda^{-2}. \tag{2.13}$$

Also, by Hölder's inequality we have

$$\int_{-1}^{1} \frac{B_\lambda(t)}{\sqrt{1-t^2}}\,dt \leqslant (\|B_\lambda\|_{L^1})^{3/7}\,(\|B_\lambda\|_{L^\infty})^{4/7}\left(\int_{-1}^{1}(1-t^2)^{-7/8}\,dt\right)^{4/7} \leqslant C_0\lambda^{-4/7}. \tag{2.14}$$

Hence $C_{B_\lambda, s} < \infty$, where $C_{B,s}$ is given in Lemma 2.1, and so Lemma 2.1 may be applied.

Let $F_i \in P_2(\mathbf{R}^3; 0, 1)$, $i = 1, 2, 3, 4$, and let Q^+, $Q^{\delta+}$, Q^+_λ, $Q^{\delta+}_\lambda$ be the collision gain operators with the kernels $B(\langle z/|z|, \sigma\rangle)$, $B(\langle z/|z|, \sigma\rangle)\,1_{\{|z| \geqslant \delta\}}$, $B_\lambda(\langle z/|z|, \sigma\rangle)$ and $B_\lambda(\langle z/|z|, \sigma\rangle)\,1_{\{|z| \geqslant \delta\}}$ respectively, where $z = v - v_*$ as above. Let

$$G(v) = Q^+(F_1, F_2)(v), \quad G_\delta(v) = Q^{\delta+}(F_1, F_2)(v), \quad G_{\lambda,\delta}(v) = Q^{\delta+}_\lambda(F_1, F_2)(v),$$

$$H(v) = Q^+(F_3, F_4)(v), \quad H_\delta(v) = Q^{\delta+}(F_3, F_4)(v), \quad H_{\lambda,\delta}(v) = Q^{\delta+}_\lambda(F_3, F_4)(v).$$

Then

$$\begin{aligned}
&\|[(F_1 \circ F_2) \circ (F_3 \circ F_4)] * \phi_\varepsilon - (F_1 \circ F_2) \circ (F_3 \circ F_4)\|_{\mathscr{B}_0} \\
&\quad = \|Q^+(G, H) * \phi_\varepsilon - Q^+(G, H)\|_{\mathscr{B}_0} \leqslant 2\,\|Q^+_\lambda(G_{\lambda,\delta}, H_{\lambda,\delta}) - Q^+(G, H)\|_{\mathscr{B}_0} \\
&\qquad + \|Q^+_\lambda(G_{\lambda,\delta}, H_{\lambda,\delta}) * \phi_\varepsilon - Q^+_\lambda(G_{\lambda,\delta}, H_{\lambda,\delta})\|_{\mathscr{B}_0}
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
&\|Q^+_\lambda(G_{\lambda,\delta}, H_{\lambda,\delta}) - Q^+(G, H)\|_{\mathscr{B}_0} \\
&\quad \leqslant \|Q^+_\lambda(G_{\lambda,\delta}, H_{\lambda,\delta}) - Q^+(G_{\lambda,\delta}, H_{\lambda,\delta})\|_{\mathscr{B}_0} + \|Q^+(G_{\lambda,\delta}, H_{\lambda,\delta}) - Q^+(G_\delta, H_\delta)\|_{\mathscr{B}_0} \\
&\qquad + \|Q^+(G_\delta, H_\delta) - Q^+(G, H)\|_{\mathscr{B}_0}.
\end{aligned}$$

Since $F_i \in P_2(\mathbf{R}^3; 0, 1)$, we have

$$\|G\|_{\mathscr{B}_0}, \quad \|G_\delta\|_{\mathscr{B}_0}, \quad \|G_{\lambda,\delta}\|_{\mathscr{B}_0}, \quad \|H\|_{\mathscr{B}_0}, \quad \|H_\delta\|_{\mathscr{B}_0}, \quad \|H_{\lambda,\delta}\|_{\mathscr{B}_0} \leqslant 1.$$

Using the formula (1.17),

$$\|Q_\lambda^+(G_{\lambda,\delta}, H_{\lambda,\delta}) - Q^+(G_{\lambda,\delta}, H_{\lambda,\delta})\|_{\mathscr{B}_0} \leqslant 2\pi\, \|B_\lambda - B\|_{\mathscr{B}_0} \leqslant 2\pi\Omega_B(\lambda),$$
$$\|Q^+(G_{\lambda,\delta}, H_{\lambda,\delta}) - Q^+(G_\delta, H_\delta)\|_{\mathscr{B}_0} \leqslant \|G_{\lambda,\delta} - G_\delta\|_{\mathscr{B}_0} + \|H_{\lambda,\delta} - H_\delta\|_{\mathscr{B}_0} \leqslant 4\pi\Omega_B(\lambda),$$
$$\|Q^+(G_\delta, H_\delta) - Q^+(G, H)\|_{\mathscr{B}_0} \leqslant \|G_\delta - G\|_{\mathscr{B}_0} + \|H_\delta - H\|_{\mathscr{B}_0}$$

and for $\|G_\delta - G\|_{\mathscr{B}_0}$, $\|H_\delta - H\|_{\mathscr{B}_0}$ we have for instance

$$\|G_\delta - G\|_{\mathscr{B}_0} = \|Q^{\delta+}(F_1, F_2) - Q^+(F_1, F_2)\|_{\mathscr{B}_0} \leqslant \iint_{\mathbf{R}^3\times\mathbf{R}^3} 1_{\{|v-v_*|<\delta\}}\, dF_1(v)\, dF_2(v_*).$$

Suppose, for instance, $|||F_1 - M|||_0 \leqslant q$. Then by Lemma 2.2,

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3} 1_{\{|v-v_*|<\delta\}}\, dF_1(v)\, dF_2(v_*)$$
$$\leqslant \|F_2\|_{\mathscr{B}_0} \sup_{v_*\in\mathbf{R}^3} \int_{|v-v_*|<\delta} dF_1(v) \leqslant C_0(\delta^{-2}\, |||F_1 - M|||_0 + \delta^3) \leqslant C_0(\delta^{-2}q + \delta^3).$$

The same argument also holds for $\|H_\delta - H\|_{\mathscr{B}_0}$, and therefore we get

$$\|G_\delta - G\|_{\mathscr{B}_0} + \|H_\delta - H\|_{\mathscr{B}_0} \leqslant C_0(\delta^{-2}q + \delta^3).$$

In summary,

$$2\,\|Q_\lambda^+(G_{\lambda,\delta}, H_{\lambda,\delta}) - Q^+(G, H)\|_{\mathscr{B}_0} \leqslant 12\pi\Omega_B(\lambda) + C_0(\delta^{-2}q + \delta^3). \qquad (2.16)$$

Next, we estimate $\|Q_\lambda^+(G_{\lambda,\delta}, H_{\lambda,\delta}) * \phi_\varepsilon - Q_\lambda^+(G_{\lambda,\delta}, H_{\lambda,\delta})\|_{\mathscr{B}_0}$. Since B_λ is smooth, for any $0 < s < 1/2$, Lemma 2.1 provides

$$Q_\lambda^+(G_{\lambda,\delta}, H_{\lambda,\delta}) \equiv Q_\lambda^+(Q_\lambda^{\delta+}(F_1, F_2), Q_\lambda^{\delta+}(F_3, F_4)) \in \mathscr{B}_2(\mathbf{R}^3) \cap H^s(\mathbf{R}^3)$$

and, by (2.13) and (2.14),

$$\|Q_\lambda^+(G_{\lambda,\delta}, H_{\lambda,\delta})\|_{H^s} \leqslant \frac{C_0}{\sqrt{1-2s}} (\|B_\lambda\|_{1,\infty})^2\, \|B_\lambda\|_{L^1([-1,1],\frac{dt}{\sqrt{1-t^2}})} \delta^{-2}$$
$$\leqslant \frac{C_0}{\sqrt{1-2s}}\, \lambda^{-4-4/7}\delta^{-2}.$$

Thus, by part (ii) of Lemma 2.2 with $s = 3/7$, and the fact that $\|Q_\lambda^+(H_{\lambda,\delta}, G_{\lambda,\delta})\|_{\mathscr{B}_2} \leqslant 4$,

$$\|Q_\lambda^+(G_{\lambda,\delta}, H_{\lambda,\delta}) * \phi_\varepsilon - Q_\lambda^+(G_{\lambda,\delta}, H_{\lambda,\delta})\|_{\mathscr{B}_0}$$
$$\leqslant C_0(\|Q_\lambda^+(G_{\lambda,\delta}, H_{\lambda,\delta})\|_{H^s})^{4/7}\, \varepsilon^{4s/7} \leqslant C_0(\lambda^{-4-4/7}\delta^{-2})^{4/7}\, \varepsilon^{12/49}.$$

Together with (2.15) and (2.16), this implies that for all $0<\delta\leqslant 1$ and $0<\lambda\leqslant 2$

$$\|[(F_1\circ F_2)\circ(F_3\circ F_4)]*\phi_\varepsilon-(F_1\circ F_2)\circ(F_3\circ F_4)\|_{\mathscr{B}_0}$$
$$\leqslant C_0(\Omega_B(\lambda)+\delta^{-2}q+\delta^3+(\lambda^{-4-4/7}\delta^{-2})^{4/7}\,\varepsilon^{12/49}).$$

Now choose $\lambda=q^{1/60}$, $\delta=(q/2)^{1/180}$, then, recalling that $\varepsilon=q^{2/7-1/120}$ and $0<q\leqslant 2$, we compute

$$\|[(F_1\circ F_2)\circ(F_3\circ F_4)]*\phi_\varepsilon-(F_1\circ F_2)\circ(F_3\circ F_4)\|_{\mathscr{B}_0}$$
$$\leqslant C_0(\Omega_B(q^{1/60})+q^{1/60}+q^{\frac{12}{49}(\frac{2}{7}-\frac{1}{120})-\frac{1}{60}\cdot\frac{440}{147}}),$$

and $\frac{12}{49}(\frac{2}{7}-\frac{1}{120})-\frac{1}{60}\cdot\frac{440}{147}>\frac{1}{60}$. Then by the property of $\Omega_B(\cdot)$ and $0<q\leqslant 2$ we have

$$q^{1/60}+q^{\frac{12}{49}(\frac{2}{7}-\frac{1}{120})-\frac{1}{60}\cdot\frac{440}{147}}\leqslant 3q^{1/60}\leqslant 12\pi\Omega_B(q^{1/60}).\quad\blacksquare$$

Lemma 2.5. Let $F\in P_2(\mathbf{R}^3;0,1)$. Then for any $0<\delta<1$,

$$\|Q_n^+(F)-M_F\|_{\mathscr{B}_0}\leqslant C_0(n^{-\delta}+\Omega_B(q_n^{1/60})),\qquad n=1,2,3,... \tag{2.17}$$

where

$$q_n=\max_{n^{1-\delta}\leqslant m\leqslant n}|||Q_m^+(F)-M_F|||_0$$

and C_0 is an absolute constant.

Proof. Throughtout the proof, we simply denote M_F by M. Let $N=\lceil n^{1-\delta}\rceil$ be the least integer not less than $n^{1-\delta}$. If $n\leqslant 4N$, then $1\leqslant 4N/n\leqslant 4(n^{1-\delta}+1)/n\leqslant 8n^{1-\delta}/n=8n^{-\delta}$ and so

$$\|Q_n^+(F)-M\|_{\mathscr{B}_0}\leqslant 2\leqslant 16n^{-\delta}.$$

So in the following we can assume that $n>4N$. Also whenever $Q_n^+(F)-M$ is considered, we can assume that $q_n>0$. Let $\phi_\varepsilon(z)=(1/\varepsilon)^3\,\phi_1(z/\varepsilon)$ be the function used in the proof of Lemma 2.2, here and below $\varepsilon=q_n^{2/7-1/120}$. (Recall that $q_n\leqslant 2$). Consider

$$\|Q_n^+(F)-M\|_{\mathscr{B}_0}\leqslant\|Q_n^+(F)-Q_n^+(F)*\phi_\varepsilon\|_{\mathscr{B}_0}$$
$$+\|(Q_n^+(F)-M)*\phi_\varepsilon\|_{\mathscr{B}_0}+\|M*\phi_\varepsilon-M\|_{\mathscr{B}_0}.$$

For the first term, consider the following decomposition (recall definition of $Q_n^+(F)$):

$$Q_n^+(F) * \phi_\varepsilon - Q_n^+(F) = [P_n(F) * \phi_\varepsilon - P_n(F)] + [R_n(F) * \phi_\varepsilon - R_n(F)] \tag{2.18}$$

where (using $f \circ g = g \circ f$)

$$P_n(F) = \frac{1}{n-1} \sum_{k=2N}^{n-2N} Q_k^+(F) \circ Q_{n-k}^+(F),$$

$$R_n(F) = \frac{2}{n-1} \sum_{k=1}^{2N-1} Q_k^+(F) \circ Q_{n-k}^+(F).$$

For the principle term $P_n(F)$, consider further iteration

$$Q_k^+(F) \circ Q_{n-k}^+(F)$$
$$= \frac{1}{(k-1)(n-k-1)} \sum_{i=1}^{k-1} \sum_{j=1}^{n-k-1} [Q_i^+(F) \circ Q_{k-i}^+(F)] \circ [Q_j^+(F) \circ Q_{n-k-j}^+(F)]$$

which gives

$$\|P_n(F) * \phi_\varepsilon - P_n(F)\|_{\mathscr{B}_0}$$
$$\leqslant \frac{1}{n-1} \sum_{k=2N}^{n-2N} \frac{1}{(k-1)(n-k-1)} \sum_{i=1}^{k-1} \sum_{j=1}^{n-k-1}$$
$$\times \|\{[Q_i^+(F) \circ Q_{k-i}^+(F)] \circ [Q_j^+(F) \circ Q_{n-k-j}^+(F)]\} * \phi_\varepsilon$$
$$- [Q_i^+(F) \circ Q_{k-i}^+(F)] \circ [Q_j^+(F) \circ Q_{n-k-j}^+(F)]\|_{\mathscr{B}_0}.$$

For any $k \in \{2N,..., n-2N\}$, any $i \in \{1,..., k-1\}$ and any $j \in \{1,..., n-k-1\}$, we have: either $i \geqslant k/2 \geqslant N$ or $k-i \geqslant k/2 \geqslant N$; and either $j \geqslant (n-k)/2 \geqslant N$ or $n-k-j \geqslant (n-k)/2 \geqslant N$. This implies that for any $k \in \{2N,..., n-2N\}$

$$\min\{|||Q_i^+(F) - M|||_0, |||Q_{k-i}^+(F) - M|||_0\} \leqslant q_n, \qquad i = 1,..., k-1;$$
$$\min\{|||Q_j^+(F) - M|||_0, |||Q_{n-k-j}^+(F) - M|||_0\} \leqslant q_n, \qquad j = 1,..., n-k-1.$$

Since $Q_k^+(F) \in P_2(\mathbf{R}^3; 0, 1)$ $\forall k \in \mathbf{N}$, it follows from Lemma 2.4 that, with $\varepsilon = q_n^{2/7-1/120}$,

$$\max_{1 \leqslant i \leqslant k-1, 1 \leqslant j \leqslant n-k-1} \|\{[Q_i^+(F) \circ Q_{k-i}^+(F)] \circ [Q_j^+(F) \circ Q_{n-k-j}^+(F)]\} * \phi_\varepsilon$$
$$- [Q_i^+(F) \circ Q_{k-i}^+(F)] \circ [Q_j^+(F) \circ Q_{n-k-j}^+(F)]\|_{\mathscr{B}_0} \leqslant C_0 \Omega_B(q_n^{1/60})$$

for all $k = 2N, \ldots, n-2N$. Thus

$$\|P_n(F) * \phi_\varepsilon - P_n(F)\|_{\mathscr{B}_0} \leqslant C_0 \Omega_B(q_n^{1/60}), \qquad \varepsilon = q_n^{2/7-1/120}.$$

The estimate for the remainder $R_n(F)$ is easy:

$$\begin{aligned} &\|R_n(F) * \phi_\varepsilon - R_n(F)\|_{\mathscr{B}_0} \\ &\quad \leqslant 2\,\|R_n(F)\|_{\mathscr{B}_0} = \frac{4}{n-1} \sum_{k=1}^{2N-1} \|Q_k^+(F) \circ Q_{n-k}^+(F)\|_{\mathscr{B}_0} = \frac{4}{n-1}(2N-1) \leqslant C_0 n^{-\delta}. \end{aligned}$$

Thus by (2.18)

$$\|Q_n^+(F) * \phi_\varepsilon - Q_n^+(F)\|_{\mathscr{B}_0} \leqslant C_0(n^{-\delta} + \Omega_B(q_n^{1/60})). \tag{2.19}$$

For the second term, using Lemma 2.2 part (i) and notice that $|||Q_n^+(F) - M|||_0 \leqslant q_n$, we have

$$\begin{aligned} \|(Q_n^+(F) - M) * \phi_\varepsilon\|_{\mathscr{B}_0} &\leqslant C_0 \varepsilon^{-2} (|||Q_n^+(F) - M|||_0)^{4/7} \\ &\leqslant C_0 q_n^{-4/7+1/60} q_n^{4/7} = C_0 q_n^{1/60} \leqslant C_0 4\pi \Omega_B(q_n^{1/60}). \end{aligned} \tag{2.20}$$

Finally for the third term,

$$\|M * \phi_\varepsilon - M\|_{\mathscr{B}_0} \leqslant \int_{|z| \leqslant 1} \phi_1(z) \left(\int_{\mathbf{R}^3} |M(v - \varepsilon z) - M(v)|\, dv \right) dz,$$

using the elementary inequality $|e^y - 1| \leqslant |y|\, e^{|y|}$ for $y \in \mathbf{R}$, we have for all $|z| \leqslant 1$

$$|M(v - \varepsilon z) - M(v)| \leqslant C_0 \varepsilon e^{-|v|^2/2}(1 + |v|)\, e^{2|v|}.$$

So

$$\|M * \phi_\varepsilon - M\|_{\mathscr{B}_0} \leqslant C_0 \varepsilon \leqslant C_0 q_n^{1/60} \leqslant C_0 4\pi \Omega_B(q_n^{1/60}).$$

This together with (2.19) and (2.20) gives the estimate (2.17). ∎

3. ESTIMATE OF $|||Q_n^+(F) - M_F|||_0$

For any $\alpha \geqslant 0$ and $H \in \mathscr{B}_2(\mathbf{R}^3)$ with vanishing total mass and sufficiently many vanishing moments, $|||H|||_\alpha$ is given by

$$|||H|||_\alpha = \sup_{|\xi| > 0} \frac{|\hat{H}(\xi)|}{|\xi|^{2+\alpha}}.$$

These norms have been introduced into kinetic theory by Gabetta, Toscani, and Wennberg in ref. 16. For Maxwellian molecules they have a contraction property, which, for $\alpha = 0$, is equivalent to the contraction property first discovered by Tanaka.[25, 26]

For any constant $L > 0$ we introduce the following semi-norms $|||\cdot|||_\alpha^{(\leqslant L)}$ and $|||\cdot|||_\alpha^{(\geqslant L)}$:

$$|||H|||_\alpha^{(\leqslant L)} = \sup_{0<|\xi|\leqslant L} \frac{|\hat{H}(\xi)|}{|\xi|^{2+\alpha}}, \qquad |||H|||_\alpha^{(\geqslant L)} = \sup_{|\xi|\geqslant L} \frac{|\hat{H}(\xi)|}{|\xi|^{2+\alpha}}.$$

It is obvious that $|||H|||_\alpha = \max\{|||H|||_\alpha^{(\leqslant L)}, |||H|||_\alpha^{(\geqslant L)}\}$. For any $F \in \mathscr{B}_2(\mathbf{R}^3)$, let

$$p_{i,j}(F) = \int_{\mathbf{R}^3} (v_i v_j - \tfrac{1}{3}|v|^2 \delta_{i,j})\, dF(v), \qquad i, j = 1, 2, 3.$$

Here v_i are components of $v = (v_1, v_2, v_3)$ and $\delta_{i,j} = 1$ for $i = j$; $= 0$ for $i \neq j$. For a constant $L > 0$, let χ_L be a C_c^∞-function on $\mathbf{R}^3$ satisfying

$$0 \leqslant \chi_L \leqslant 1 \quad \text{on} \quad \mathbf{R}^3; \quad \chi_L(\xi) = 1 \quad \text{for} \quad |\xi| \leqslant L.$$

Then define a linear transformation $F \mapsto P_F\colon \mathscr{B}_2(\mathbf{R}^3) \to \mathscr{B}_0(\mathbf{R}^3)$ through the Fourier transform:

$$\widehat{P_F}(\xi) = -\tfrac{1}{2}\left(\sum_{i,j} p_{i,j}(F)\, \xi_i \xi_j\right) \chi_L(\xi), \qquad \xi \in \mathbf{R}^3.$$

i.e.,

$$dP_F(v) = \left[-\tfrac{1}{2} \sum_{i,j} p_{i,j}(F) (2\pi)^{-3} \int_{\mathbf{R}^3} \xi_i \xi_j \chi_L(\xi)\, e^{\mathrm{i}\langle v, \xi\rangle}\, d\xi \right] dv.$$

Referring to ref. 12, we now define the following functional on $P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_s(\mathbf{R}^3)$ for $s > 2$:

$$\Phi^*(F) = \max\{|||F - P_F - M|||_\alpha^{(\leqslant L)}\, L^\alpha + K\, |||P_F|||_0,\ |||F - M|||_0^{(\geqslant L)}\} \tag{3.1}$$

where $K \geqslant 1$ is a constant, and $\alpha = (s-2)/(s-1)$, which implies that $\Phi^*(F) < \infty$ for $F \in P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_s(\mathbf{R}^3)$ (see also below).

The functions $p_{i,j}(F)$ have the following contraction property (refs. 16 and 12):

$$p_{i,j}(F \circ G) = \frac{a}{2} [p_{i,j}(F) + p_{i,j}(G)], \qquad F, G \in P_2(\mathbf{R}^3; 0, 1) \tag{3.2}$$

where

$$a = \tfrac{1}{4} + \tfrac{3}{4} \cdot 2\pi \int_0^\pi B(\cos(\theta)) \sin(\theta) \cos^2(\theta)\, d\theta < 1. \tag{3.3}$$

Lemma 3.1. Let $s > 2$ be a constant, $\alpha = (s-2)/(s-1)$. Then there are finite constants $L > 0$, $K \geqslant 1$, $0 < b < 1$ and $A > 0$ which depend only on the kernel $B(\cdot)$ and s, such that

$$\Phi^*(F \circ G) \leqslant \frac{1-b}{2} [\Phi^*(F) + \Phi^*(G)] \qquad \forall F, G \in P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_s(\mathbf{R}^3)$$

and

$$|||F - M|||_0 \leqslant \Phi^*(F) \leqslant A\, \|F\|_{\mathscr{B}_s}^{1/(s-1)} \qquad \forall F \in P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_s(\mathbf{R}^3)$$

where Φ^* is the functional given in (3.1) with the constants α, L and K.

Proof. We first prove the second estimate. Let $F \in P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_s(\mathbf{R}^3)$. We have

$$|||F - M|||_0^{(\leqslant L)} \leqslant |||F - P_F - M|||_\alpha^{(\leqslant L)} L^\alpha + |||P_F|||_0.$$

Since $|||F - M|||_0 = \max\{|||F - M|||_0^{(\leqslant L)}, |||F - M|||_0^{(\geqslant L)}\}$ and $K \geqslant 1$, it follows that

$$|||F - M|||_0 \leqslant \Phi^*(F).$$

On the other hand, by definition of $p_{i,j}(F)$ and P_F, we have

$$\widehat{P_F}(\xi) = -\tfrac{1}{2}\left(\int_{\mathbf{R}^3} [\langle v, \xi\rangle^2 - \tfrac{1}{3} |v|^2 |\xi|^2]\, dF(v)\right) \chi_L(\xi)$$
$$= -\tfrac{1}{2}\left(\int_{\mathbf{R}^3} \langle v, \xi\rangle^2\, dF(v) - \int_{\mathbf{R}^3} \langle v, \xi\rangle^2 M(v)\, dv\right) \chi_L(\xi)$$

This gives

$$|||P_F|||_0 \leqslant 1 \tag{3.4}$$

and

$$\widehat{P_F}(\xi) = -\tfrac{1}{2} \int_{\mathbf{R}^3} \langle v, \xi\rangle^2\, dF(v) + \tfrac{1}{2} |\xi|^2, \qquad |\xi| \leqslant L \tag{3.5}$$

and then using (2.12) we get for $|\xi| \leqslant L$

$$\hat{F}(\xi) - \widehat{P_F}(\xi) - \hat{M}(\xi) = -\int_0^1 (1-\tau) \int_{\mathbf{R}^3} \langle v, \xi\rangle^2 \left[e^{-i\langle \tau\xi, v\rangle} - 1\right] d[F-M](v)\, d\tau.$$

In the inequality (2.9) choosing $\alpha = (s-2)/(s-1)$, $h = \tau\,|\xi|\,|v|$ and using the Hölder inequality we obtain that for any $H \in \mathscr{B}_s(\mathbf{R}^3)$

$$\left|\int_{\mathbf{R}^3} \langle v, \xi\rangle^2 \left[e^{-i\langle \tau\xi, v\rangle} - 1\right] dH(v)\right| \leqslant 2^{1-\alpha}\, |\xi|^{2+\alpha}\, \|H\|_{\mathscr{B}_s}^{1/(s-1)}\, \|H\|_{\mathscr{B}_2}^{\alpha}, \quad 0 \leqslant \tau \leqslant 1.$$

Applying this inequality to $H = F - M$ gives

$$|||F - P_F - M|||_{\alpha}^{(\leqslant L)} \leqslant 2^{-\alpha}\, \|F-M\|_{\mathscr{B}_s}^{1/(s-1)}\, \|F-M\|_{\mathscr{B}_2}^{\alpha}. \tag{3.6}$$

On the other hand by the inequality (2.11), we have $|||F-M|||_0^{(\geqslant L)} \leqslant |||F-M|||_0 \leqslant 2$. Since $4 = \|F\|_{\mathscr{B}_2} \leqslant \|F\|_{\mathscr{B}_s}$, and since $\|M\|_{\mathscr{B}_s}$ depends only on s, it follows that

$$1 \leqslant \|F\|_{\mathscr{B}_s}^{1/(s-1)}, \qquad \text{and} \qquad \|F\|_{\mathscr{B}_s}^{1/(s-1)} + \|M\|_{\mathscr{B}_s}^{1/(s-1)} \leqslant C_s\, \|F\|_{\mathscr{B}_s}^{1/(s-1)}.$$

It then follows from (3.4)–(3.6) that

$$\Phi^*(F) \leqslant \left(\|F\|_{\mathscr{B}_s}^{1/(s-1)} + \|M\|_{\mathscr{B}_s}^{1/(s-1)}\right) 2^{2\alpha} L^{\alpha} + K + 2 \leqslant A\, \|F\|_{\mathscr{B}_s}^{1/(s-1)}$$

where the constant A depends only on s, L and K. This proves the second estimate in the lemma.

Next we prove the first estimate. Let $F, G \in P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_s(\mathbf{R}^3)$. Then $F \circ G \in P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_s(\mathbf{R}^3)$ so that $\Phi^*(F \circ G)$ makes sense. Consider the decomposition:

$$\begin{aligned} F \circ G - P_{F\circ G} - M = {} & (F - P_F) \circ (G - P_G) - M + F \circ P_G \\ & + P_F \circ G - P_F \circ P_G - P_{F\circ G} \end{aligned}$$

which gives

$$\begin{aligned} |||F \circ G - P_{F\circ G} - M|||_{\alpha}^{(\leqslant L)} \leqslant {} & |||(F - P_F) \circ (G - P_G) - M|||_{\alpha}^{(\leqslant L)} \\ & + |||F \circ P_G + P_F \circ G - P_F \circ P_G - P_{F\circ G}|||_{\alpha}^{(\leqslant L)}. \end{aligned} \tag{3.7}$$

Furthermore, since $M \circ M = M$,

$$\begin{aligned} & (F - P_F) \circ (G - P_G) - M \\ & \quad = \tfrac{1}{2}(F - P_F - M) \circ (G - P_G + M) + \tfrac{1}{2}(F - P_F + M) \circ (G - P_G - M), \end{aligned}$$

and then, using the Bobylev identity (2.1),

$$[(F-P_F)\circ(G-P_G)]\hat{}\,(\xi)-\hat{M}(\xi)$$

$$=\tfrac{1}{2}\int_{\mathbf{S}^2} B(\langle\xi/|\xi|,\sigma\rangle)[\hat{F}(\xi_+)-\widehat{P_F}(\xi_+)-\hat{M}(\xi_+)][\hat{G}(\xi_-)-\widehat{P_G}(\xi_-)+\hat{M}(\xi_-)]\,d\sigma$$

$$+\tfrac{1}{2}\int_{\mathbf{S}^2} B(\langle\xi/|\xi|,\sigma\rangle)[\hat{F}(\xi_+)-\widehat{P_F}(\xi_+)+\hat{M}(\xi_+)][\hat{G}(\xi_-)-\widehat{P_G}(\xi_-)-\hat{M}(\xi_-)]\,d\sigma.$$

Recall that $|\xi_+|=\cos(\theta/2)\,|\xi|$, $|\xi_-|=\sin(\theta/2)\,|\xi|$, and notice that $|\widehat{P_F}(\xi_\pm)| \leqslant |||P_F|||_0\,|\xi_\pm|^2\leqslant|\xi_\pm|^2$, $|\widehat{P_G}(\xi_\pm)|\leqslant|||P_G|||_0\,|\xi_\pm|^2\leqslant|\xi_\pm|^2$. We have for $|\xi|\leqslant L$

$$\begin{aligned}
&|[\hat{F}(\xi_+)-\widehat{P_F}(\xi_+)-\hat{M}(\xi_+)][\hat{G}(\xi_-)-\widehat{P_G}(\xi_-)+\hat{M}(\xi_-)]|\\
&\qquad\leqslant |||F-P_F-M|||_\alpha\,\cos^{2+\alpha}(\theta/2)\,|\xi|^{2+\alpha}\,[2+L^2\sin^2(\theta/2)],\\
&|[\hat{F}(\xi_+)-\widehat{P_F}(\xi_+)+\hat{M}(\xi_+)][\hat{G}(\xi_-)-\widehat{P_G}(\xi_-)-\hat{M}(\xi_-)]|\\
&\qquad\leqslant |||G-P_G-M|||_\alpha\,\sin^{2+\alpha}(\theta/2)\,|\xi|^{2+\alpha}\,[2+L^2\cos^2(\theta/2)].
\end{aligned}$$

Since $\bar{B}(\theta):=2\pi B(\cos(\theta))\sin(\theta)=\bar{B}(\pi-\theta)$, it follows that

$$|[(F-P_F)\circ(G-P_G)]\hat{}\,(\xi)-\hat{M}(\xi)|/|\xi|^{2+\alpha}$$

$$\leqslant\int_{\mathbf{R}^3} B(\langle\xi/|\xi|,\sigma\rangle)\Bigg(|||F-P_F-M|||_\alpha^{(\leqslant L)}\cos^{2+\alpha}(\theta/2)\left[1+\frac{1}{2}L^2\sin^2(\theta/2)\right]$$

$$+|||G-P_G-M|||_\alpha^{(\leqslant L)}\sin^{2+\alpha}(\theta/2)\left[1+\frac{1}{2}L^2\cos^2(\theta/2)\right]\Bigg)\,d\sigma$$

$$=\frac{\lambda_1(L)}{2}\left(|||F-P_F-M|||_\alpha^{(\leqslant L)}+|||G-P_G-M|||_\alpha^{(\leqslant L)}\right)$$

where

$$\begin{aligned}
\lambda_1(L)=\int_0^\pi \bar{B}(\theta)(\cos^{2+\alpha}(\theta/2)[1+\tfrac{1}{2}L^2\sin^2(\theta/2)]\\
+\sin^{2+\alpha}(\theta/2)[1+\tfrac{1}{2}L^2\cos^2(\theta/2)])\,d\theta.
\end{aligned}$$

Thus

$$\begin{aligned}
&|||(F-P_F)\circ(G-P_G)-M|||_\alpha^{(\leqslant L)}\\
&\qquad\leqslant\frac{\lambda_1(L)}{2}\left(|||F-P_F-M|||_\alpha^{(\leqslant L)}+|||G-P_G-M|||_\alpha^{(\leqslant L)}\right).
\end{aligned}\tag{3.8}$$

Next, we estimate $|||F\circ P_G+P_F\circ G-P_F\circ P_G-P_{F\circ G}|||_\alpha^{(\leqslant L)}$. We have

$$(F\circ P_G)^\wedge(\xi)+(P_F\circ G)^\wedge(\xi)-(P_F\circ P_G)^\wedge(\xi)-\widehat{P_{F\circ G}}(\xi)$$

$$=\int_{\mathbf{S}^2}B(\langle\xi/|\xi|,\sigma\rangle)([\hat{F}(\xi_+)-1]\,\widehat{P_G}(\xi_-)$$

$$+\widehat{P_F}(\xi_+)[\hat{G}(\xi_-)-1]-\widehat{P_F}(\xi_+)\,\widehat{P_G}(\xi_-))\,d\sigma$$

$$+\int_{\mathbf{S}^2}B(\langle\xi/|\xi|,\sigma\rangle)[\widehat{P_F}(\xi_+)+\widehat{P_G}(\xi_-)]\,d\sigma-\widehat{P_{F\circ G}}(\xi),\qquad |\xi|\leqslant L.$$

Note that the last difference is zero, i.e.,

$$\widehat{P_{F\circ G}}(\xi)=\int_{\mathbf{S}^2}B(\langle\xi/|\xi|,\sigma\rangle)[\widehat{P_F}(\xi_+)+\widehat{P_G}(\xi_-)]\,d\sigma,\qquad |\xi|\leqslant L.\tag{3.9}$$

In fact, applying the formula (2.2) (i.e., exchanging the positions of $\langle(v-v_*)/|v-v_*|,\sigma\rangle$ and $\langle\xi/|\xi|,\sigma\rangle$), and recalling that F, G have zero mean,

$$\int_{\mathbf{R}^3}\langle v,\xi\rangle^2\,d(F\circ G)(v)$$

$$=\int_{\mathbf{S}^2}B(\langle\xi/|\xi|,\sigma\rangle)\left(\int_{\mathbf{R}^3}\langle v,\xi_+\rangle^2\,dF(v)+\int_{\mathbf{R}^3}\langle v_*,\xi_-\rangle^2\,dG(v_*)\right)d\sigma.$$

Then applying (3.5) to $F\circ G$, F and G respectively and using $|\xi|^2\equiv|\xi_+|^2+|\xi_-|^2$ we obtain (3.9). Thus, for all $0<|\xi|\leqslant L$ we have

$$|(F\circ P_G)^\wedge(\xi)+(P_F\circ G)^\wedge(\xi)-(P_F\circ P_G)^\wedge(\xi)-\widehat{P_{F\circ G}}(\xi)|$$

$$\leqslant\int_{\mathbf{S}^2}B(\langle\xi/|\xi|,\sigma\rangle)(|\hat{F}(\xi_+)-1|\,|\widehat{P_G}(\xi_-)|$$

$$+|\widehat{P_F}(\xi_+)|\,|\hat{G}(\xi_-)-1|+|\widehat{P_F}(\xi_+)\,\widehat{P_G}(\xi_-)|)\,d\sigma.$$

Since $|\hat{F}(\xi)-1|\leqslant\frac{3}{2}|\xi|^2$ (see (2.12)), $|\widehat{P_F}(\xi)|\leqslant|||P_F|||_0\,|\xi|^2$, $|||P_F|||_0\leqslant 1$, and the same is true for G, it follows that

$$|(F\circ P_G)^\wedge(\xi)+(P_F\circ G)^\wedge(\xi)-(P_F\circ P_G)^\wedge(\xi)-\widehat{P_{F\circ G}}(\xi)|$$

$$\leqslant(\tfrac{3}{2}|||P_F|||_0+\tfrac{3}{2}|||P_G|||_0+|||P_F|||_0\,|||P_G|||_0)\int_{\mathbf{S}^2}B(\langle\xi/|\xi|,\sigma\rangle)\,|\xi_+|^2\,|\xi_-|^2\,d\sigma$$

$$\leqslant\tfrac{1}{2}(|||P_F|||_0+|||P_G|||_0)\,|\xi|^4,\qquad |\xi|\leqslant L.$$

Therefore,

$$\|\!|F \circ P_G + P_F \circ G - P_F \circ P_G - P_{F \circ G}|\!\|_\alpha^{(\leqslant L)} \leqslant \tfrac{1}{2}\,(\|\!|P_F|\!\|_0 + \|\!|P_G|\!\|_0)\, L^{2-\alpha}$$

which, together with (3.7) and (3.8), yields

$$\begin{aligned}&\|\!|F \circ G - P_{F \circ G} - M|\!\|_\alpha^{(\leqslant L)}\, L^\alpha \\ &\leqslant \frac{\lambda_1(L)}{2}\,(\|\!|F - P_F - M|\!\|_\alpha^{(\leqslant L)}\, L^\alpha + \|\!|G - P_G - M|\!\|_\alpha^{(\leqslant L)}\, L^\alpha) + \frac{L^2}{2}\,(\|\!|P_F|\!\|_0 + \|\!|P_G|\!\|_0).\end{aligned}$$

Also, by the formula (3.2) we have

$$\widehat{P_{F \circ G}}\,(\xi) = \frac{a}{2}\,[\widehat{P_F}\,(\xi) + \widehat{P_G}\,(\xi)], \qquad \|\!|P_{F \circ G}|\!\|_0 \leqslant \frac{a}{2}\,[\|\!|P_F|\!\|_0 + \|\!|P_G|\!\|_0]$$

which, together with the above inequality, gives

$$\begin{aligned}\|\!|F \circ G - P_{F \circ G} - M|\!\|_\alpha^{(\leqslant L)}\, L^\alpha + K\, \|\!|P_{F \circ G}|\!\|_0 &\\ \leqslant \frac{\lambda_1(L)}{2}\,(\|\!|F - P_F - M|\!\|_\alpha^{(\leqslant L)}\, L^\alpha + \|\!|G - P_G - M|\!\|_\alpha^{(\leqslant L)}\, L^\alpha)&\\ + \frac{L^2 + Ka}{2}\,(\|\!|P_F|\!\|_0 + \|\!|P_G|\!\|_0).&\end{aligned} \tag{3.10}$$

It remains to estimate $\|\!|F \circ G - M|\!\|_0^{(\geqslant L)}$. Write, as above,

$$\begin{aligned}&\hat{F}(\xi_+)\, \hat{G}(\xi_-) - \hat{M}(\xi_+)\, \hat{M}(\xi_-) \\ &\quad = \tfrac{1}{2}[\hat{F}(\xi_+) - \hat{M}(\xi_+)][\hat{G}(\xi_-) + \hat{M}(\xi_-)] + \tfrac{1}{2}[\hat{G}(\xi_-) - \hat{M}(\xi_-)][\hat{F}(\xi_+) + \hat{M}(\xi_+)].\end{aligned}$$

Since $\hat{M}(\xi) = e^{-|\xi|^2/2}$, $|\xi_+| = |\xi| \cos(\theta/2)$, $|\xi_-| = |\xi| \sin(\theta/2)$, we have

$$|\xi| \geqslant L \Rightarrow \hat{M}(\xi_+) \leqslant e^{-\frac{L^2}{2}\cos^2(\theta/2)}, \qquad \hat{M}(\xi_-) \leqslant e^{-\frac{L^2}{2}\sin^2(\theta/2)}.$$

Hence by $|\hat{F}(\xi_\pm)|$, $|\hat{G}(\xi_\pm)| \leqslant 1$ and $\bar{B}(\theta) = \bar{B}(\pi - \theta)$ we have for all $|\xi| \geqslant L$

$$\begin{aligned}&\frac{|(F \circ G)^\wedge\,(\xi) - \hat{M}(\xi)|}{|\xi|^2} \\ &\qquad \leqslant \frac{1}{2}\int_{S^2} B(\langle \xi/|\xi|, \sigma\rangle)\left(\frac{|\hat{F}(\xi_+) - \hat{M}(\xi_+)|}{|\xi_+|^2}\,[1 + \hat{M}(\xi_-)]\cos^2(\theta/2)\right.\\ &\qquad\quad \left. + \frac{|\hat{G}(\xi_-) - \hat{M}(\xi_-)|}{|\xi_-|^2}\,[1 + \hat{M}(\xi_+)]\sin^2(\theta/2)\right) d\sigma\end{aligned}$$

$$\leqslant \frac{1}{2}\int_0^\pi \bar{B}(\theta)(|||F-M|||_0\,[1+e^{-\frac{L^2}{2}\sin^2(\theta/2)}]\cos^2(\theta/2)$$
$$+|||G-M|||_0\,[1+e^{-\frac{L^2}{2}\cos^2(\theta/2)}]\sin^2(\theta/2))\,d\sigma$$
$$=\frac{\lambda_2(L)}{2}\,[|||F-M|||_0+|||G-M|||_0]$$

where

$$\lambda_2(L)=\int_0^\pi \bar{B}(\theta)\,(\tfrac{1}{2}\,[1+e^{-\frac{L^2}{2}\sin^2(\theta/2)}]\cos^2(\theta/2)+\tfrac{1}{2}\,[1+e^{-\frac{L^2}{2}\cos^2(\theta/2)}]\sin^2(\theta/2))\,d\theta.$$

Thus,

$$|||F\circ G-M|||_0^{(\geqslant L)}\leqslant \frac{\lambda_2(L)}{2}\,[|||F-M|||_0+|||G-M|||_0],$$

and since $|||F-M|||_0\leqslant \Phi^*(F)$ we obtain

$$|||F\circ G-M|||_0^{(\geqslant L)}\leqslant \frac{\lambda_2(L)}{2}\,[\Phi^*(F)+\Phi^*(G)]. \tag{3.11}$$

Note that by definition of $\lambda_1(L)$ we can choose a small $L>0$ such that $\lambda_1(L)<1$. Since $0<\lambda_2(L)$ and $a<1$, we can choose $b>0$ such that

$$\max\{\lambda_1(L),\,\lambda_2(L),\,a\}<1-b.$$

We then choose the constant K such that

$$1\leqslant K \qquad \text{and} \qquad \frac{L^2+Ka}{1-b}\leqslant K,\quad \text{e.g.,}\quad K=\max\left\{1,\,\frac{L^2}{1-a-b}\right\}.$$

By (3.10), (3.11), and by definition of $\Phi^*(\cdot)$, we then have

$$|||F\circ G-P_{F\circ G}-M|||_\alpha^{(\leqslant L)}\,L^\alpha+K\,|||P_{F\circ G}|||_0$$
$$\leqslant \frac{1-b}{2}\left(|||F-P_F-M|||_\alpha^{(\leqslant L)}\,L^\alpha+\frac{L^2+Ka}{1-b}\,|||P_F|||_0\right.$$
$$\left.+|||G-P_G-M|||_\alpha^{(\leqslant L)}\,L^\alpha+\frac{L^2+Ka}{1-b}\,|||P_G|||_0\right)\leqslant \frac{1-b}{2}\,[\Phi^*(F)+\Phi^*(G)].$$

and

$$|||F \circ G - M|||_0^{(\geqslant L)} \leqslant \frac{1-b}{2}\,[\Phi^*(F) + \Phi^*(G)].$$

Therefore

$$\Phi^*(F \circ G) \leqslant \frac{1-b}{2}\,[\Phi^*(F) + \Phi^*(G)].$$

Finally we note that the constants L, K, and b depend only on $B(\cdot)$ and $\alpha = (s-2)/(s-1)$. ▮

Lemma 3.2. Let $0 < c < 1$ be a constant and a_n be real numbers with the relation

$$0 \leqslant a_n \leqslant \frac{c}{n-1} \sum_{k=1}^{n-1} a_k, \qquad n = 2, 3, \ldots$$

Then

$$a_n \leqslant c \left(\frac{2}{n}\right)^{1-c} a_1, \qquad n = 2, 3, \ldots \tag{3.12}$$

Proof. Consider

$$A_n = \frac{1}{n-1} \sum_{k=1}^{n-1} a_k, \qquad n = 2, 3, \ldots$$

By assumption we have $a_n \leqslant cA_n$, $n = 2, 3, \ldots$ This implies for any $n \geqslant 2$,

$$A_{n+1} = \frac{1}{n} \sum_{k=1}^{n} a_k \leqslant \frac{1}{n}\,[(n-1)\,A_n + cA_n] = \left(1 - \frac{b}{n}\right) A_n \leqslant \left(\frac{n}{n+1}\right)^b A_n$$

where $b = 1 - c > 0$. This implies that $A_n \leqslant (\frac{2}{n})^b\, A_2$, $n \geqslant 2$. Since $a_n \leqslant cA_n$ and $A_2 = a_1$, this gives the inequality (3.12). ▮

Lemma 3.3. Let $s > 2$ be a constant and let $0 < b < 1$ be the constant obtained in Lemma 3.1. Then there is a constant $0 < C < \infty$ which depends only on the kernel $B(\cdot)$ and s, such that for all $F \in P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_s(\mathbf{R}^3)$

$$|||Q_n^+(F) - M|||_0 \leqslant C\, \|F\|_{\mathscr{B}_s}^{1/(s-1)} \left(\frac{1}{n}\right)^b, \qquad n = 1, 2, 3, \ldots$$

Proof. Let Φ^* be the functional given by (3.1) and let $F \in P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_s(\mathbf{R}^3)$. Since $Q_n^+(F) \in P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_s(\mathbf{R}^3)$, $\Phi^*(Q_n^+(F))$ makes sense. By Lemma 3.1, (1.10) and notice that the functional $F \mapsto P_F$ is linear, $P_{Q_n^+(F)} = \frac{1}{n-1} \sum_{k=1}^{n-1} P_{Q_k^+(F) \circ Q_{n-k}^+(F)}$, we have

$$\Phi^*(Q_n^+(F)) \leqslant \frac{1}{n-1} \sum_{k=1}^{n-1} \Phi^*(Q_k^+(F) \circ Q_{n-k}^+(F))$$
$$\leqslant \frac{1-b}{n-1} \sum_{k=1}^{n-1} \Phi^*(Q_k^+(F)), \quad n = 2, 3, \ldots$$

Therefore by Lemma 3.2 (with $c = 1-b$) and Lemma 3.1

$$\Phi^*(Q_n^+(F)) \leqslant (1-b)\left(\frac{2}{n}\right)^b \Phi^*(F) \leqslant (1-b)\, 2^b A\, \|F\|_{\mathscr{B}_s}^{1/(s-1)} \left(\frac{1}{n}\right)^b, \qquad n \geqslant 2.$$

Thus with $C = 2^b A$ which depends only on $B(\cdot)$ and s and using Lemma 3.1 again we obtain $|||Q_n^+(F) - M|||_0 \leqslant \Phi^*(Q_n^+(F)) \leqslant C\, \|F\|_{\mathscr{B}_s}^{1/(s-1)}\, n^{-b}$, $n = 1, 2, 3, \ldots$ ∎

A natural connection between Lemma 2.5 and Lemma 3.3 is given by the following lemma:

Lemma 3.4. Let $F \in P_2(\mathbf{R}^3; 0, 1)$. Then for any $0 < \varepsilon < \infty$, there is an $F_\varepsilon \in P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_4(\mathbf{R}^3)$ such that

$$\|F_\varepsilon\|_{\mathscr{B}_4} \leqslant 280(1+3\varepsilon)^5 \left(\int_{\mathbf{R}^3} |v|\, dF(v)\right)^{-4} \cdot \frac{1}{\varepsilon} \tag{3.13}$$

and

$$|||F_\varepsilon - F|||_0 \leqslant 15(\varepsilon^{1/3} + J_F(\varepsilon^{1/3})). \tag{3.14}$$

where $J_F(r)$ is defined in (1.20).

Proof. Given any $\varepsilon > 0$. By Riesz representation theorem, there is a unique measure $F_\varepsilon \in \mathscr{B}_0$ such that

$$\int_{\mathbf{R}^3} \phi(v)\, dF_\varepsilon(v) = \lambda \int_{\mathbf{R}^3} \frac{\phi(\frac{v}{\rho} - v_\varepsilon)}{1 + \varepsilon\, |v|^2}\, dF(v) \tag{3.15}$$

for all $\phi \in C_b(\mathbf{R}^3)$, where $\lambda = \lambda_\varepsilon > 0$, $\rho = \rho_\varepsilon > 0$ and $v_\varepsilon \in \mathbf{R}^3$ are determined uniquely by the system of equations

$$\lambda \int_{\mathbf{R}^3} \frac{(1, \frac{v}{\rho} - v_\varepsilon, |\frac{v}{\rho} - v_\varepsilon|^2)}{1+\varepsilon\,|v|^2}\, dF(v) = (1, 0, 3) \tag{3.16}$$

which is solved as follows:

$$\lambda = \frac{1}{\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\, dF(v)}, \qquad v_\varepsilon = \frac{1}{\rho} \cdot \frac{\int_{\mathbf{R}^3} \frac{v}{1+\varepsilon\,|v|^2}\, dF(v)}{\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\, dF(v)}$$

$$\rho^2 = \frac{(\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\, dF(v))(\int_{\mathbf{R}^3} \frac{|v|^2}{1+\varepsilon\,|v|^2}\, dF(v)) - |\int_{\mathbf{R}^3} \frac{v}{1+\varepsilon\,|v|^2}\, dF(v)|^2}{3(\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\, dF(v))^2}.$$

Note that by Cauchy–Schwarz inequality, the right hand side of the last equality is strictly positive, as it must be for this to define ρ.

This shows that the function F_ε satisfying (3.15)–(3.16) is well defined. Moreover since $F \in P_2(\mathbf{R}^3; 0, 1)$, it easily seen that $F_\varepsilon \in P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_4(\mathbf{R}^3)$ and the equality (3.15) holds for all Borel functions ϕ satisfying $\sup_{v \in \mathbf{R}^3} |\phi(v)|\, (1+|v|^2)^{-2} < \infty$.

We now prove (3.13) and (3.14). The first step is to estimate λ, ρ, v_ε.

For any real number y, using $\int_{\mathbf{R}^3} v\, dF(v) = 0$ we have

$$\begin{aligned} &\left| \int_{\mathbf{R}^3} \frac{v}{1+\varepsilon\,|v|^2}\, dF(v) \right|^2 \\ &\quad = \left| \int_{\mathbf{R}^3} \frac{(1 - y(1+\varepsilon\,|v|^2))\, v}{1+\varepsilon\,|v|^2}\, dF(v) \right|^2 \\ &\quad \leqslant \left(\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\, dF(v) - 2y + (1+3\varepsilon)\, y^2 \right) \left(\int_{\mathbf{R}^3} \frac{|v|^2}{1+\varepsilon\,|v|^2}\, dF(v) \right). \end{aligned}$$

The last term takes on its minimum at $y = 1/(1+3\varepsilon)$, and hence

$$\left| \int_{\mathbf{R}^3} \frac{v}{1+\varepsilon\,|v|^2}\, dF(v) \right|^2 \leqslant \left(\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\, dF(v) - \frac{1}{1+3\varepsilon} \right) \left(\int_{\mathbf{R}^3} \frac{|v|^2}{1+\varepsilon\,|v|^2}\, dF(v) \right). \tag{3.17}$$

This implies that

$$\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\, dF(v) \geqslant \frac{1}{1+3\varepsilon}$$

and

$$\left(\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\,dF(v)\right)\left(\int_{\mathbf{R}^3} \frac{|v|^2}{1+\varepsilon\,|v|^2}\,dF(v)\right) - \left|\int_{\mathbf{R}^3} \frac{v}{1+\varepsilon\,|v|^2}\,dF(v)\right|^2$$
$$\geqslant \frac{1}{1+3\varepsilon}\int_{\mathbf{R}^3} \frac{|v|^2}{1+\varepsilon\,|v|^2}\,dF(v) > 0,$$

and hence we obtain

$$1 \leqslant \lambda = \frac{1}{\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\,dF(v)} \leqslant 1+3\varepsilon \tag{3.18}$$

and

$$\rho^2 \geqslant \frac{1}{3(1+3\varepsilon)}\left(\int_{\mathbf{R}^3} \frac{|v|^2}{1+\varepsilon\,|v|^2}\,dF(v)\right)\left(\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\,dF(v)\right)^{-2}. \tag{3.19}$$

The last inequality implies that

$$\frac{1}{\rho^2} \leqslant 3(1+3\varepsilon)\left(\int_{\mathbf{R}^3} \frac{|v|^2}{1+\varepsilon\,|v|^2}\,dF(v)\right)^{-1}. \tag{3.20}$$

Also,

$$|v_\varepsilon| \leqslant 3\sqrt{\varepsilon}\,. \tag{3.21}$$

In fact, by the estimates (3.17), (3.19), and $\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\,dF(v) \leqslant 1$, we have

$$|v_\varepsilon|^2 = \frac{1}{\rho^2}\cdot\frac{|\int_{\mathbf{R}^3} \frac{v}{1+\varepsilon\,|v|^2}\,dF(v)|^2}{(\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\,dF(v))^2} \leqslant 3(1+3\varepsilon)\left(\int_{\mathbf{R}^3} \frac{1}{1+\varepsilon\,|v|^2}\,dF(v) - \frac{1}{1+3\varepsilon}\right) \leqslant 9\varepsilon.$$

Now we prove (3.13). We have

$$\|F_\varepsilon\|_{\mathscr{B}_4} = 7 + \int_{\mathbf{R}^3} |v|^4\,dF_\varepsilon(v) = 7 + \lambda\int_{\mathbf{R}^3} \frac{|\frac{v}{\rho} - v_\varepsilon|^4}{1+\varepsilon\,|v|^2}\,dF(v),$$

$$\left|\frac{v}{\rho} - v_\varepsilon\right|^4 \leqslant 8(|v|^4\,\rho^{-4} + |v_\varepsilon|^4), \qquad \int_{\mathbf{R}^3} \frac{|v|^4}{1+\varepsilon\,|v|^2}\,dF(v) \leqslant \frac{3}{\varepsilon}.$$

Hence by the estimates (3.18), (3.20), and (3.21),

$$\int_{\mathbf{R}^3} |v|^4 \, dF_\varepsilon(v) \leqslant 8\lambda \int_{\mathbf{R}^3} (|v|^4 \rho^{-4} + |v_\varepsilon|^4) \frac{1}{1+\varepsilon |v|^2} \, dF(v)$$

$$\leqslant 8(1+3\varepsilon) \left((3(1+3\varepsilon))^2 \left(\int_{\mathbf{R}^3} \frac{|v|^2}{1+\varepsilon |v|^2} \, dF(v) \right)^{-2} \cdot \frac{3}{\varepsilon} + 3^4 \varepsilon^2 \right).$$

Furthermore, using the Cauchy–Schwarz inequality,

$$\left(\int_{\mathbf{R}^3} \frac{|v|^2}{1+\varepsilon |v|^2} \, dF(v) \right)^{-1} \leqslant (1+3\varepsilon) \left(\int_{\mathbf{R}^3} |v| \, dF(v) \right)^{-2}, \quad 3^{-2} \leqslant \left(\int_{\mathbf{R}^3} |v| \, dF(v) \right)^{-4}$$

with the consequence that

$$\int_{\mathbf{R}^3} |v|^4 \, dF_\varepsilon(v) \leqslant 8(1+3\varepsilon)(3^3(1+3\varepsilon)^4 + 3^6 \varepsilon^3) \left(\int_{\mathbf{R}^3} |v| \, dF(v) \right)^{-4} \cdot \frac{1}{\varepsilon}.$$

Therefore, we compute

$$\|F_\varepsilon\|_{\mathscr{B}_4} \leqslant [7 \cdot 3^2 \varepsilon + 8(1+3\varepsilon)(3^3(1+3\varepsilon)^4 + 3^6 \varepsilon^3)] \left(\int_{\mathbf{R}^3} |v| \, dF(v) \right)^{-4} \cdot \frac{1}{\varepsilon}$$

$$\leqslant 280(1+3\varepsilon)^5 \left(\int_{\mathbf{R}^3} |v| \, dF(v) \right)^{-4} \cdot \frac{1}{\varepsilon}.$$

This proves (3.13). (The constant 280 is correct, though arriving to it requires close and careful estimation. However, the value of this constant is not crucial for the argument that follows, and the reader may substitute more expeditionous estimation if desired.)

Next, note that by $F_\varepsilon, F \in P_2(\mathbf{R}^3; 0, 1)$ we have $|||F_\varepsilon - F|||_0 \leqslant 3$. This implies that if $\varepsilon \geqslant 1/9$, then $|||F_\varepsilon - F|||_0 \leqslant 3 < 15\varepsilon^{1/3}$. So in the following we assume that $0 < \varepsilon < 1/9$. We need to estimate $\int_{\mathbf{R}^3} \frac{\varepsilon |v|^2}{1+\varepsilon |v|^2} |v|^2 \, dF(v)$ and $|\rho^2 - 1|$. We have

$$\int_{\mathbf{R}^3} \frac{\varepsilon |v|^2}{1+\varepsilon |v|^2} |v|^2 \, dF(v) \leqslant \frac{\varepsilon^{1/3}}{1+\varepsilon^{1/3}} \int_{|v| \leqslant \varepsilon^{-1/3}} |v|^2 \, dF(v) + \int_{|v| > \varepsilon^{-1/3}} |v|^2 \, dF(v)$$

$$\leqslant 3\varepsilon^{1/3} + J_F(\varepsilon^{1/3}) \leqslant 3(\varepsilon^{1/3} + J_F(\varepsilon^{1/3})). \tag{3.22}$$

This gives

$$\int_{\mathbf{R}^3} \frac{|v|^2}{1+\varepsilon |v|^2} \, dF(v) = 3 - \int_{\mathbf{R}^3} \frac{\varepsilon |v|^2}{1+\varepsilon |v|^2} |v|^2 \, dF(v) \geqslant 3 - 3(\varepsilon^{1/3} + J_F(\varepsilon^{1/3}))$$

So by (3.19) we obtain (since $0<\varepsilon<1/9$)

$$\rho^2 \geqslant \frac{1}{1+3\varepsilon}\cdot\frac{1}{3}\int_{\mathbf{R}^3}\frac{|v|^2}{1+\varepsilon\,|v|^2}\,dF(v)$$
$$\geqslant (1-3\varepsilon)(1-(\varepsilon^{1/3}+J_F(\varepsilon^{1/3}))) \geqslant 1-2(\varepsilon^{1/3}+J_F(\varepsilon^{1/3}))$$

On the other hand, by the equation that defines ρ^2,

$$\rho^2 \leqslant \frac{\int_{\mathbf{R}^3}\frac{|v|^2}{1+\varepsilon\,|v|^2}\,dF(v)}{3\int_{\mathbf{R}^3}\frac{1}{1+\varepsilon\,|v|^2}\,dF(v)} \leqslant \frac{3}{3\int_{\mathbf{R}^3}\frac{1}{1+\varepsilon\,|v|^2}\,dF(v)} = \lambda \leqslant 1+3\varepsilon \quad (<4/3).$$

Thus,

$$|\rho^2-1| \leqslant 2(\varepsilon^{1/3}+J_F(\varepsilon^{1/3})). \tag{3.23}$$

Also, by (3.21) and $\varepsilon<1/9$ we have

$$|v_\varepsilon| \leqslant 3\sqrt{\varepsilon} \leqslant 3(\varepsilon^{1/3}+J_F(\varepsilon^{1/3}))^{3/2} \qquad \text{and} \qquad |v_\varepsilon| \leqslant 1. \tag{3.24}$$

Now, to prove (3.14), consider

$$\frac{|\widehat{F_\varepsilon}(\xi)-\hat{F}(\xi)|}{|\xi|^2} \leqslant \frac{|\widehat{F_\varepsilon}(\xi)-\widehat{F_\varepsilon}(\rho\xi)|}{|\xi|^2} + \frac{|\widehat{F_\varepsilon}(\rho\xi)-\hat{F}(\xi)|}{|\xi|^2}. \tag{3.25}$$

For the first term, use the fact that for any $H\in P_2(\mathbf{R}^3; 0, 1)$

$$\left|\frac{d}{dt}(\hat{H}(t\xi))\right| \leqslant \int_0^t \left(\int_{\mathbf{R}^3} |v|^2\,|\xi|^2\,dH(v)\right)d\tau = 3\,|\xi|^2\,t, \qquad t\geqslant 0.$$

Taking $H=F_\varepsilon$ and using (3.23) gives

$$|\widehat{F_\varepsilon}(\xi)-\widehat{F_\varepsilon}(\rho\xi)| \leqslant 3\,|\xi|^2\left|\int_\rho^1 t\,dt\right| = 3\,|\xi|^2\,|\tfrac{1}{2}(1-\rho^2)| \leqslant 3(\varepsilon^{1/3}+J_F(\varepsilon^{1/3}))\,|\xi|^2.$$

Therefore

$$\frac{|\widehat{F_\varepsilon}(\xi)-\widehat{F_\varepsilon}(\rho\xi)|}{|\xi|^2} \leqslant 3(\varepsilon^{1/3}+J_F(\varepsilon^{1/3})), \qquad \xi\in\mathbf{R}^3\setminus\{0\}. \tag{3.26}$$

For the second term, we prove that

$$\frac{|\widehat{F_\varepsilon}(\rho\xi)-\hat{F}(\xi)|}{|\xi|^2} \leqslant 12(\varepsilon^{1/3}+J_F(\varepsilon^{1/3})), \qquad \xi\in\mathbf{R}^3\setminus\{0\}. \tag{3.27}$$

This estimate holds for $|\xi| \geqslant \frac{1}{\sqrt{6}}(\varepsilon^{1/3}+J_F(\varepsilon^{1/3}))^{-1/2}$ because $\|\widehat{F_\varepsilon}\|_{L^\infty}$, $\|\hat{F}\|_{L^\infty} \leqslant 1$. Now suppose $0<|\xi| \leqslant \frac{1}{\sqrt{6}}(\varepsilon^{1/3}+J_F(\varepsilon^{1/3}))^{-1/2}$. Using (2.12),

$$-\frac{\widehat{F_\varepsilon}(\rho\xi)-\hat{F}(\xi)}{|\xi|^2}$$

$$=\int_0^1 (1-\tau)\left(\int_{\mathbf{R}^3} \langle v, \rho\zeta\rangle^2 e^{-i\langle \tau\rho\xi,\, v\rangle}\, dF_\varepsilon(v)-\int_{\mathbf{R}^3} \langle v, \zeta\rangle^2 e^{-i\langle \tau\xi,\, v\rangle}\, dF(v)\right) d\tau$$

where $\zeta=\xi/|\xi|$. Now apply (3.15) with $\phi(v)=\langle v, \rho\zeta\rangle^2 e^{-i\langle \tau\rho\xi,\, v\rangle}$. Then

$$-\frac{\widehat{F_\varepsilon}(\rho\xi)-\hat{F}(\xi)}{|\xi|^2}$$

$$=\int_0^1 (1-\tau)\left(\lambda \int_{\mathbf{R}^3} \frac{\langle v-\rho v_\varepsilon, \zeta\rangle^2}{1+\varepsilon\,|v|^2}\, e^{-i\langle \tau\xi,\, v-\rho v_\varepsilon\rangle}\, dF(v)-\int_{\mathbf{R}^3} \langle v, \zeta\rangle^2 e^{-i\langle \tau\xi,\, v\rangle}\, dF(v)\right) d\tau$$

$$=\int_0^1 (1-\tau)(\lambda-1) \int_{\mathbf{R}^3} \frac{\langle v-\rho v_\varepsilon, \zeta\rangle^2}{1+\varepsilon\,|v|^2}\, e^{-i\langle \tau\xi,\, v-\rho v_\varepsilon\rangle}\, dF(v)\, d\tau$$

$$+\int_0^1 (1-\tau) \int_{\mathbf{R}^3} \left(\frac{\langle v-\rho v_\varepsilon, \zeta\rangle^2-\langle v, \zeta\rangle^2}{1+\varepsilon\,|v|^2}\right) e^{-i\langle \tau\xi,\, v-\rho v_\varepsilon\rangle}\, dF(v)\, d\tau$$

$$+\int_0^1 (1-\tau) \int_{\mathbf{R}^3} \langle v, \zeta\rangle^2 \left(\frac{1}{1+\varepsilon\,|v|^2}-1\right) e^{-i\langle \tau\xi,\, v-\rho v_\varepsilon\rangle}\, dF(v)\, d\tau$$

$$+\int_0^1 (1-\tau) \int_{\mathbf{R}^3} \langle v, \zeta\rangle^2 \left(e^{-i\langle \tau\xi,\, v-\rho v_\varepsilon\rangle}-e^{-i\langle \tau\xi,\, v\rangle}\right) dF(v)\, d\tau$$

$$:= I_1+I_2+I_3+I_4.$$

Estimating I_1: By (3.18), $\rho^2 \leqslant 4/3$ and $|v_\varepsilon| \leqslant 1$ we have

$$|I_1| \leqslant \frac{3\varepsilon}{2} \int_{\mathbf{R}^3} (\langle v, \zeta\rangle-\langle \rho v_\varepsilon, \zeta\rangle)^2\, dF(v)=\frac{3\varepsilon}{2}\left(\int_{\mathbf{R}^3} \langle v, \zeta\rangle^2\, dF(v)+\langle \rho v_\varepsilon, \zeta\rangle^2\right)$$

$$\leqslant \frac{3\varepsilon}{2}(3+\rho^2\,|v_\varepsilon|^2) \leqslant \frac{13}{2}\,\varepsilon^{2/3}\cdot \varepsilon^{1/3} \leqslant 2(\varepsilon^{1/3}+J_F(\varepsilon^{1/3})).$$

Estimating I_2: By (3.21) and $\rho^2 \leqslant 4/3$ we compute

$$|I_2| \leqslant \tfrac{1}{2} \int_{\mathbf{R}^3} (2\rho\,|v_\varepsilon|\,|v|+\rho^2\,|v_\varepsilon|^2)\, dF(v)=\rho\,|v_\varepsilon| \int_{\mathbf{R}^3} |v|\, dF(v)+\tfrac{1}{2}\,\rho^2\,|v_\varepsilon|^2$$

$$\leqslant 6\sqrt{\varepsilon}+6\varepsilon \leqslant 6\varepsilon^{1/3} \leqslant 6(\varepsilon^{1/3}+J_F(\varepsilon^{1/3})).$$

Estimating I_3: By the estimate (3.22) we have

$$|I_3| \leqslant \frac{1}{2} \int_{\mathbf{R}^3} \frac{\varepsilon\,|v|^2}{1+\varepsilon\,|v|^2}\,|v|^2\,dF(v) \leqslant \frac{3}{2}\,(\varepsilon^{1/3} + J_F(\varepsilon^{1/3})).$$

Estimating I_4: By $|\xi| \leqslant \frac{1}{\sqrt{6}}\,(\varepsilon^{1/3} + J_F(\varepsilon^{1/3}))^{-1/2}$, $\rho \leqslant 2/\sqrt{3}$ and using the inequality in (3.24),

$$\begin{aligned} |I_4| &\leqslant |\xi|\,\rho\,|v_\varepsilon| \left(\int_0^1 (1-\tau)\,\tau\,d\tau \right) \int_{\mathbf{R}^3} |v|^2\,dF(v) = |\xi|\,\rho\,|v_\varepsilon| \cdot \frac{1}{2} \\ &\leqslant \frac{1}{\sqrt{2}}\,(\varepsilon^{1/3} + J_F(\varepsilon^{1/3}))^{-1/2} \cdot (\varepsilon^{1/3} + J_F(\varepsilon^{1/3}))^{3/2} = \frac{1}{\sqrt{2}}\,(\varepsilon^{1/3} + J_F(\varepsilon^{1/3})). \end{aligned}$$

Thus, for $0 < |\xi| \leqslant \frac{1}{\sqrt{6}}\,(\varepsilon^{1/3} + J_F(\varepsilon^{1/3}))^{-1/2}$ we obtain

$$\frac{|\widehat{F_\varepsilon}\,(\rho\xi) - \hat{F}(\xi)|}{|\xi|^2} \leqslant \sum_{k=1}^{4} |I_k| \leqslant 12(\varepsilon^{1/3} + J_F(\varepsilon^{1/3})).$$

This proves (3.27). Combining (3.25)–(3.27) gives us (3.14). ∎

4. UPPER BOUNDS ON THE CONVERGENCE RATE

In this section we prove Theorem 1, Theorem 2, and the Corollary to Theorem 2. We begin by introducing the affine transform of distributions. This is necessary so that we may apply the results of the previous sections to distributions that do not necessarily belong to $P_2(\mathbf{R}^3; 0, 1)$.

Let $F \in \mathscr{B}_2(\mathbf{R}^3)$. Let $v_0 \in \mathbf{R}^3$ and $T > 0$. By Riesz Representation Theorem, we can define an affine transform F_T of F by the following equality

$$\int_{\mathbf{R}^3} \phi(v)\,dF_T(v) = \int_{\mathbf{R}^3} \phi\left(\frac{v - v_0}{\sqrt{T}}\right) dF(v) \tag{4.1}$$

for all $\phi \in C_b(\mathbf{R}^3)$. Since $F \in \mathscr{B}_2(\mathbf{R}^3)$, the equality (4.1) can be extended to all Borel function ϕ satisfying $\sup_{v \in \mathbf{R}^3} |\phi(v)|\,(1+|v|^2)^{-1} < \infty$. It is easy to check that the following properties hold:

$$\begin{aligned} &F, G \in P_2(\mathbf{R}^3; v_0, T) \Rightarrow F_T \in P_2(\mathbf{R}^3; 0, 1) \quad \text{and} \quad (F \circ G)_T = F_T \circ G_T, \\ &F \in P_2(\mathbf{R}^3; v_0, T) \Rightarrow (Q_n^+(F))_T = Q_n^+(F_T), \quad n = 1, 2, 3,\ldots. \end{aligned} \tag{4.2}$$

The norm $\|\cdot\|_{\mathscr{B}_0}$ defined in (1.14) with $s=0$ is invariant under this affine transform: $\|F\|_{\mathscr{B}_0}=\|F_T\|_{\mathscr{B}_0}$. But this invariance does not hold for the norm $\|\cdot\|_{\mathscr{B}_2}$. However we have the following inequality

$$\|F\|_{\mathscr{B}_2}\leqslant(1+2T+2\,|v_0|^2)\,\|F_T\|_{\mathscr{B}_2}\qquad \forall F\in\mathscr{B}_2(\mathbf{R}^3). \tag{4.3}$$

Similarly, if $F\geqslant 0$, then using (4.1) with $\phi(v)=|v|^2\,1_{|v|>\frac{1+|v_0|\,r}{\sqrt{T}\,r}}$ we have

$$J_{F_T}\left(\frac{\sqrt{T}\,r}{1+|v_0|\,r}\right)\leqslant\frac{2(1+|v_0|^2)}{T}\,J_F(r),\qquad 0<r\leqslant 1. \tag{4.4}$$

Another property which will be used in this section is the following variant of Tanaka's fundamental non-expansion property

$$|||Q_n^+(F)-Q_n^+(G)|||_0\leqslant|||F-G|||_0,\qquad F,G\in P_2(\mathbf{R}^3;v_0,T),\quad n=1,2,\dots. \tag{4.5}$$

This in turn is a consequence of the fact that if $F_1, F_2, G_1, G_2\in P_2(\mathbf{R}^3)$ have the same mean, then

$$|||F_1\circ G_1-F_2\circ G_2|||_0\leqslant\tfrac{1}{2}(|||F_1-F_2|||_0+|||G_1-G_2|||_0).$$

This is much easier to prove for the norm $|||\cdot|||_0$, using the Bobylev identity, than in Tanaka's original setting.

The following simple lemma plays an important role in the proof; it gives us convergence in $\|\cdot\|_{\mathscr{B}_2}$ essentially for free, once we have it in $\|\cdot\|_{\mathscr{B}_0}$.

Lemma 4.1. For all G in $P_2(\mathbf{R}^3;0,1)$,

$$\|G-M\|_{\mathscr{B}_2}\leqslant C_0\,\|G-M\|_{\mathscr{B}_0}\,\log\left(\frac{2e}{\|G-M\|_{\mathscr{B}_0}}\right), \tag{4.6}$$

where $M=M_G$ is the Maxwellian in $P_2(\mathbf{R}^3;0,1)$, C_0 is an absolute constant.

Proof. If $G=M$, there is nothing to prove. Otherwise, consider the identity (using the Jordan decomposition)

$$|G-M|=G-M+2(M-G)^+$$

where $(\mu)^+=\frac{1}{2}(|\mu|+\mu)$ denotes the positive part of μ. Then since G and M both belong to $P_2(\mathbf{R}^3;0,1)$ and $(M-G)^+\leqslant|G-M|$ and $(M-G)^+\leqslant M$, it follows that for any $R\geqslant 1$

$$\|G-M\|_{\mathscr{B}_2} = 2\int_{\mathbf{R}^3} (1+|v|^2)\, d(M-G)^+ (v)$$

$$\leqslant 4R\, \|G-M\|_{\mathscr{B}_0} + (2\pi)^{-3/2}\, e^{-R/4} \int_{|v|>\sqrt{R}} (1+|v|^2)\, e^{-|v|^2/4}\, dv$$

$$\leqslant 4R\, \|G-M\|_{\mathscr{B}_0} + C_0 e^{-R/4}. \tag{4.7}$$

Choose $R = 4\log(2e/\|G-M\|_{\mathscr{B}_0})$. Then $R \geqslant 1$ because $0 < \|G-M\|_{\mathscr{B}_0} \leqslant 2$. This together with (4.7) gives (4.6). ∎

It is worth noting that the proof may be applied much more generally. For the sake of simplicity, we have avoided a general formulation.

Before proving Theorem 1, let us first note that the function $r \mapsto \Phi_{B,F}(r)$ given in Theorem 1 is bounded, non-decreasing on $[0, \infty)$, continuous at $r=0$ and $\Phi_{B,F}(0) = 0$ because the functions $r \mapsto \Omega_B^*(r)$ and $r \mapsto J_F(r)$ are bounded, non-decreasing on $[0, \infty)$, continuous at $r = 0$ and $\Omega_B^*(0) = J_F(0) = 0$.

Proof of Theorem 1. The proof of the estimate (1.23) for $F \in P_2(\mathbf{R}^3; v_0, T)$ is essentially equivalent to that for the standard case $F \in P_2(\mathbf{R}^3; 0, 1)$. However since the function $J_F(\cdot)$ is not invariant under the affine transform of distributions, we have to maintain the notation F and denote by $F_T \in P_2(\mathbf{R}^3; 0, 1)$ the affine transform of F. From the affine transform (4.1) and (4.2) we have

$$(Q_n^+(F) - M_F)_T = (Q_n^+(F))_T - (M_F)_T = Q_n^+(F_T) - M.$$

where M is the Maxwellian distribution in $P_2(\mathbf{R}^3; 0, 1)$. This together with the inequality (4.3) gives

$$\|Q_n^+(F) - M_F\|_{\mathscr{B}_2} \leqslant (1+2T+2\,|v_0|^2)\, \|Q_n^+(F_T) - M\|_{\mathscr{B}_2}. \tag{4.8}$$

Next we prove that

$$\|Q_n^+(F_T) - M\|_{\mathscr{B}_0} \leqslant C^{(0)}_{B,F}\, \Omega_B((n^{-\alpha} + J_F(n^{-\alpha}))^{1/60}), \tag{4.9}$$

$$C^{(0)}_{B,F} = C_B (1+|v_0|)^{1/30}\, (1+T)^{1/20} \left(\int_{\mathbf{R}^3} |v-v_0|\, dF(v) \right)^{-1/30} \tag{4.10}$$

where the constant C_B depends only on $B(\cdot)$, but will change from line to line.

We shall apply Lemma 2.5 to the measure $F_T \in P_2(\mathbf{R}^3; 0, 1)$. We first prove that for any $\varepsilon > 0$

$$\||Q_m^+(F_T) - M\||_0 \leqslant 15(\varepsilon^{1/3} + J_{F_T}(\varepsilon^{1/3})) + C_B (C_{F_T,\varepsilon})^{1/3} \left(\frac{1}{\varepsilon}\right)^{1/3} \left(\frac{1}{m}\right)^{b} \tag{4.11}$$

where

$$C_{F_T,\varepsilon} = 280(1+3\varepsilon)^5 \left(\int_{\mathbf{R}^3} |v| \, dF_T(v) \right)^{-4} = 280(1+3\varepsilon)^5 \, T^2 \left(\int_{\mathbf{R}^3} |v-v_0| \, dF(v) \right)^{-4} \tag{4.12}$$

and here $0<b<1$ is the constant in Lemma 3.3 with $s=4$. Let $F_{T,\varepsilon} \in P_2(\mathbf{R}^3; 0, 1) \cap \mathscr{B}_4(\mathbf{R}^3)$ be the measure given in Lemma 3.4 with respect to the measure F_T and $s=4$. By (4.5), Lemma 3.3 and Lemma 3.4 we have

$$\begin{aligned} |||Q_m^+(F_T)-M|||_0 &\leqslant |||Q_m^+(F_T)-Q_m^+(F_{T,\varepsilon})|||_0 + |||Q_m^+(F_{T,\varepsilon})-M|||_0 \\ &\leqslant |||F_T - F_{T,\varepsilon}|||_0 + C_B(\|F_{T,\varepsilon}\|_{\mathscr{B}_4})^{1/3} \left(\frac{1}{m}\right)^b \\ &\leqslant 15(\varepsilon^{1/3} + J_{F_T}(\varepsilon^{1/3})) + C_B(C_{F_T,\varepsilon})^{1/3} \left(\frac{1}{\varepsilon}\right)^{1/3} \left(\frac{1}{m}\right)^b. \end{aligned}$$

This proves (4.11).

To prove (4.9), we need to balance some exponents: Suppose $\varepsilon = n^{-3\alpha}$ for some constants $\alpha>0$. According to Lemma 2.5 and the above estimate (4.11), we see that the exponents α and δ (see Lemma 2.5) are best chosen as $\alpha = \frac{1}{2}(1-\delta)\, b$, $\delta = \alpha/60$. This gives

$$\alpha = \frac{60b}{120+b}, \qquad \delta = \frac{b}{120+b}.$$

The reason will be clear from the following derivation. Let

$$q_n = \max_{n^{1-\delta} \leqslant m \leqslant n} |||Q_m^+(F_T)-M|||_0, \qquad n \geqslant 1.$$

For any integer $n \geqslant 1$, choose

$$\varepsilon = \left(\frac{\sqrt{T}\, n^{-\alpha}}{1+|v_0|\, n^{-\alpha}} \right)^3.$$

Then by the inequality (4.4) and (4.12) we have

$$J_{F_T}(\varepsilon^{1/3}) = J_{F_T}\left(\frac{\sqrt{T}\, n^{-\alpha}}{1+|v_0|\, n^{-\alpha}} \right) \leqslant \frac{2(1+|v_0|^2)}{T} J_F(n^{-\alpha}),$$

$$(C_{F_T,\varepsilon})^{1/3} \leqslant (280(1+3T^{3/2})^5)^{1/3} \, T^{2/3} \left(\int_{\mathbf{R}^3} |v-v_0| \, dF(v) \right)^{-4/3},$$

and therefore using (4.11) and $\alpha-(1-\delta)\,b=-\alpha$ we get

$$|||Q_m^+(F_T)-M|||_0 \leqslant C_{B,F}^{(1)}(n^{-\alpha}+J_F(n^{-\alpha})), \qquad m\geqslant n^{1-\delta}$$

where

$$C_{B,F}^{(1)}=C_B\left\{\sqrt{T}+\frac{(1+|v_0|^2)}{T}\right.$$
$$\left.+(1+T^{3/2})^{5/3}\,(1+|v_0|)\,T^{1/6}\left(\int_{\mathbf{R}^3}|v-v_0|\,dF(v)\right)^{-4/3}\right\}.$$

Thus

$$q_n^{1/60}\leqslant (C_{B,F}^{(1)})^{1/60}\,(n^{-\alpha}+J_F(n^{-\alpha}))^{1/60}:=C_{B,F}^{(2)}(n^{-\alpha}+J_F(n^{-\alpha}))^{1/60}.$$

By the property of $\Omega_B(\cdot)$ given in Lemma 2.3 we have, with $C_{B,F}^{(3)}=1+C_{B,F}^{(2)}$,

$$\Omega_B(q_n^{1/60})\leqslant \Omega_B(C_{B,F}^{(2)}(n^{-\alpha}+J_F(n^{-\alpha}))^{1/60})\leqslant C_{B,F}^{(3)}\Omega_B((n^{-\alpha}+J_F(n^{-\alpha}))^{1/60}).$$

Therefore, by Lemma 2.5 and $n^{-\delta}=n^{-\alpha/60}\leqslant 3\pi\Omega_B(n^{-\alpha/60})$ we obtain that for all $n\geqslant 1$

$$\|Q_n^+(F_T)-M\|_{\mathscr{B}_0}\leqslant C_0\{n^{-\delta}+\Omega_B(q_n^{1/60})\}\leqslant C_{B,F}^{(4)}\Omega_B((n^{-\alpha}+J_F(n^{-\alpha}))^{1/60})$$

where $C_{B,F}^{(4)}=C_0(3\pi+C_{B,F}^{(3)})$. Next, we compute the constants: By definition of $C_{B,F}^{(i)}$ $(i=1,2,3,4)$ we have

$$C_{B,F}^{(4)}\leqslant (3\pi+1)\,C_0$$
$$+C_0(C_B)^{1/60}\,(1+|v_0|)^{1/30}\left\{\sqrt{T}+\frac{1}{T}+\frac{(1+T^{3/2})^{5/3}\,T^{1/6}}{(\int_{\mathbf{R}^3}|v-v_0|\,dF(v))^{4/3}}\right\}^{1/60}.$$

Furthermore, by $F\in P_2(\mathbf{R}^3;v_0,T)$ we have $1/T\leqslant 3(\int_{\mathbf{R}^3}|v-v_0|\,dF(v))^{-2}$ which gives, after application of the arithmetic-geometric mean inequality to eliminate fractional powers,

$$\sqrt{T}+\frac{1}{T}+(1+T^{3/2})^{5/3}\,T^{1/6}\left(\int_{\mathbf{R}^3}|v-v_0|\,dF(v)\right)^{-4/3}$$
$$\leqslant 6\left(\int_{\mathbf{R}^3}|v-v_0|\,dF(v)\right)^{-2}(1+T)^2$$

and so $C^{(4)}_{B,F} \leqslant C_B(1+|v_0|)^{1/30}\,(1+T)^{1/20}\,(\int_{\mathbf{R}^3} |v-v_0|\,dF(v))^{-1/30} := C^{(0)}_{B,F}$. This proves (4.9).

For later estimates, we require the coefficient $C^{(0)}_{B,F}$ of $\Omega_B(\cdot)$ is not less than $2e$. In fact we have

$$(1+T)^{1/20}\left(\int_{\mathbf{R}^3} |v-v_0|\,dF(v)\right)^{-1/30} \geqslant (1+T)^{1/20}\,(3T)^{-1/60} > 1.$$

Thus if $C_B \geqslant 2e$, then $C^{(0)}_{B,F} \geqslant 2e$. Of course we can assume that $C_B \geqslant 2e$.

Now from (4.9) and $C^{(0)}_{B,F} \geqslant 2e$ and the definition of $\Omega^*_B(\cdot)$ one easily checks that

$$\|Q^+_n(F_T)-M\|_{\mathscr{B}_0}\,\log\left(\frac{2e}{\|Q^+_n(F_T)-M\|_{\mathscr{B}_0}}\right) \leqslant C^{(0)}_{B,F}\Omega^*_B((n^{-\alpha}+J_F(n^{-\alpha}))^{1/60}). \tag{4.13}$$

Combining the estimates (4.8), (4.6), (4.13), and (4.10) we obtain that for any $n \in \mathbf{N}$

$$\|Q^+_n(F)-M_F\|_{\mathscr{B}_2} \leqslant (1+2T+2\,|v_0|^2)\,C_0 C^{(0)}_{B,F}\Omega^*_B((n^{-\alpha}+J_F(n^{-\alpha}))^{1/60})$$

and $(1+2T+2\,|v_0|^2)\,C_0 C^{(0)}_{B,F} \leqslant C_B(1+|v_0|)^3\,(1+T)^2\,(\int_{\mathbf{R}^3} |v-v_0|\,dF(v))^{-1/30} := \mathrm{C}_{\mathrm{B,\,F}}$. This proves (1.23) and the proof of Theorem 1 is completed. ▮

Proof of Theorem 2. Recalling that the Wild sum (1.9) also holds for distributional solutions and using the identity $\sum_{n=1}^{\infty} e^{-t}(1-e^{-t})^{n-1} = 1$, $t \geqslant 0$ we have

$$\|f_t - M_F\|_{\mathscr{B}_2} \leqslant \sum_{n=1}^{\infty} e^{-t}(1-e^{-t})^{n-1}\,\|Q^+_n(F)-M_F\|_{\mathscr{B}_2}, \qquad t \geqslant 0.$$

Also we have

$$\|Q^+_n(F)-M_F\|_{\mathscr{B}_2} \leqslant \|Q^+_n(F)\|_{\mathscr{B}_2} + \|M_F\|_{\mathscr{B}_2} = 2(1+|v_0|^2+3T).$$

Let $\eta > 0$ satisfy

$$\frac{\eta b}{120+b} = 1-\eta, \qquad \text{i.e.,} \quad \eta = \frac{120+b}{120+2b}.$$

For any $t \geqslant 0$, choose an integer N such that $e^{\eta t} \leqslant N \leqslant 2e^{\eta t}$. Then, by Theorem 1, we have

$$\begin{aligned}\|f_t - M_F\|_{\mathscr{B}_2} &\leqslant \sum_{n=1}^{N} e^{-t}(1-e^{-t})^{n-1} \|Q_n^+(F) - M_F\|_{\mathscr{B}_2} \\ &\quad + \sum_{n=N+1}^{\infty} e^{-t}(1-e^{-t})^{n-1} \|Q_n^+(F) - M_F\|_{\mathscr{B}_2} \\ &\leqslant 2(1+|v_0|^2+3T)\, Ne^{-t} + (1-e^{-t})^N\, \Phi_{B,F}(N^{-\alpha}) \\ &\leqslant 12(1+|v_0|)^2\,(1+T)\, e^{-(1-\eta)\,t} + \Phi_{B,F}(e^{-\eta\alpha t})\end{aligned}$$

where $\alpha = 60b/(120+b)$. Since $\eta\alpha/60 = \eta b/(120+b) = 1-\eta$, the properties of $\Omega_B(\cdot)$ implies that

$$e^{-(1-\eta)\,t} = e^{-(\eta\alpha/60)\,t} \leqslant 3\pi\Omega_B^*((e^{-\eta\alpha t} + J_F(e^{-\eta\alpha t}))^{1/60}).$$

By definition of $\Phi_{B,F}(r)$ given in Theorem 1 and recalling that $\int_{\mathbf{R}^3} |v-v_0|\, dF(v) \leqslant (3T)^{1/2}$ we get

$$\begin{aligned}\|f_t - M_F\|_{\mathscr{B}_2} &\leqslant 12(1+|v_0|)^2\,(1+T)\, e^{-(1-\eta)\,t} + \Phi_{B,F}(e^{-\eta\alpha t}) \\ &\leqslant \frac{(120+C_B)(1+|v_0|)^3\,(1+T)^2}{(\int_{\mathbf{R}^3}|v-v_0|\, dF(v))^{1/30}}\, \Omega_B^*((e^{-\eta\alpha t} + J_F(e^{-\eta\alpha t}))^{1/60}) \\ &:= \tilde{\Phi}_{B,F}(e^{-\eta\alpha t}).\end{aligned}$$

Here $\tilde{\Phi}_{B,F}(r)$ is a modification of $\Phi_{B,F}(r)$ by only replacing C_B with $120+C_B$. Since $\eta\alpha = \frac{120+b}{120+2b} \cdot \frac{60b}{120+b} = \frac{30b}{60+b} = \beta$, this proves the estimate (1.24). ∎

Proof of Corollary to Theorem 2. The Hölder condition (1.25) is equivalent to $\Omega_B(r) \leqslant C_B r^\alpha$ with $0 < \alpha = \alpha_B \leqslant 1$. Let $\beta_1 = 30b/(60+b)$ be the constant in Theorem 2. Using the inequality $y\,|\log y| \leqslant 7y^{6/7}$ for $0 \leqslant y \leqslant 1$ we have $\Omega_B^*(r) \leqslant 7(\Omega_B(r))^{6/7} \leqslant 7(C_B)^{6/7} r^{6\alpha/7}$, and so by Theorem 2 and the definition of $\Phi_{B,F}(r)$

$$\begin{aligned}\|f_t - M_F\|_{\mathscr{B}_2} &\leqslant \Phi_{B,F}(e^{-\beta_1 t}) \leqslant C_{B,F}[(e^{-\beta_1 t} + J_F(e^{-\beta_1 t}))^{1/60}]^{6\alpha/7} \\ &\leqslant C_{B,F}(e^{-(\beta_1\alpha/70)\,t} + (J_F(e^{-\beta_1 t}))^{\alpha/70}), \qquad t \geqslant 0.\end{aligned}$$

Here $C_{B,F}$ denotes the finite constant that depend only on the kernel $B(\cdot)$ and the initial datum F. Furthermore by assumption on F we have for any $t \geqslant 0$

$$J_F(e^{-\beta_1 t}) \leqslant e^{-\delta\beta_1 t} \int_{\mathbf{R}^3} |v|^{2+\delta}\, dF(v) = Ce^{-\delta\beta_1 t}.$$

Let $\beta = \min\{\beta_1\alpha/70, \delta\beta_1\alpha/70\}$. Then

$$e^{-(\beta_1\alpha/70)\,t} + (J_F(e^{-\beta_1 t}))^{\alpha/70} \leqslant Ce^{-\beta t}.$$

This proves the corollary. ∎

5. LOWER BOUNDS FOR THE CONVERGENCE RATE

5.1. Construction of Initial Data

This section is devoted to the lower bounds on the convergence rate given in Theorem 4. Our starting point is the construction of solutions tending to equilibrium at an arbitrarily slow exponential rate due to A. V. Bobylev.[7] Bobylev's technique is based on the use of the Fourier transform. This method readily produces solutions in $L^1(\mathbf{R}^3)$ that decay at an arbitraily slow exponential rate:

$$\|f_t - M_F\|_{L^1} \geqslant \|\hat{f}_t - \widehat{M_F}\|_{L^\infty} \geqslant ce^{-\delta t}, \qquad t \geqslant 0 \tag{5.1}$$

where $c > 0$ is a constant and $\delta > 0$ can be arbitrarily small.

A key point in the proof is to show that this can be done in such a manner that the solutions are non-negative, and therefore physically relevant. The initial data Bobylev used was built out of an eigenfunction of the linearized Boltzmann equation in the isotropic sector. As long as one uses a single eigenfunction, exponentially decaying lower bounds are the best that one can hope to obtain.

Here, we build our initial data out of linear combinations of such eigenfunctions to obtain slower convergence. To motivate this, recall a theorem of S. Bernstein (see ref. 15) about completely monotone functions: *A function $A(t)$ is completey monotone on $[0, \infty)$ if and only if there is a probability measure ν on $[0, \infty)$ such that*

$$A(t) = \int_0^\infty e^{-st}\,d\nu(s), \qquad t \geqslant 0; \quad \nu(\{0\}) = 0. \tag{5.2}$$

Note that here the condition $\nu(\{0\}) = 0$ is equivalent to the condition $\lim_{t\to\infty} A(t) = 0$. Then, since there are completely monotone functions that decay very slowly (see Lemma 5.4 below), Bernstein's theorem suggests that we seek appropriate averages of Bobylev's initial data.

Again, the challenge is to show that one can do this in such a way that one produces a non-negative, physical solution. Our strategy for this is rather different for Bobylev's, which worked well for single eigenfunctions.

We prove a comparison property between solutions of the linearized equation and the full non-linear equation which reduces the positivity question to a simple question about the linearized equation. The investigation is carried out in three subsections, with this one focused on constructing the initial data.

Let $A(\cdot)$ be a completely monotone function on $[0, \infty)$. We first find a positive Borel measure μ on $[0, 1]$ such that for some constants $1/2 \leqslant a < \infty$ and $c > 0$

$$cA(t/a) \leqslant \int_0^1 e^{-st}\, d\mu(s) \leqslant A(t/a), \qquad \forall t \geqslant 0. \tag{5.3}$$

Let ν be the probability measure on $[0, \infty)$ determined by $A(\cdot)$ through (5.2). Choose a constant $1/2 \leqslant a < \infty$ such that $\nu([0, a]) > 0$. By the Riesz representation theorem, there is a unique positive Borel measure μ on $[0, 1]$, such that

$$\int_0^1 \phi(s)\, d\mu(s) = \int_0^a \phi\left(\frac{s}{a}\right) d\nu(s), \qquad \forall \phi \in C([0, 1]).$$

Choose $\phi(s) = e^{-st}$ for each fixed $t \geqslant 0$. This gives

$$A(t/a) \geqslant \int_0^a e^{-(s/a)\, t}\, d\nu(s) = \int_0^1 e^{-st}\, d\mu(s), \qquad \forall t \geqslant 0.$$

On the other hand, by the inequality

$$\int_{(a, \infty)} e^{-(s/a)\, t}\, d\nu(s) \leqslant e^{-t} \nu((a, \infty)) \leqslant \frac{\nu((a, \infty))}{\nu([0, a])} \int_0^a e^{-(s/a)\, t}\, d\nu(s)$$

we have

$$A(t/a) \leqslant \left(1 + \frac{\nu((a, \infty))}{\nu([0, a])}\right) \int_0^a e^{-(s/a)\, t}\, d\nu(s) = \frac{1}{\nu([0, a])} \int_0^1 e^{-st}\, d\mu(s).$$

This proves inequality (5.3) with the constant $c = \nu([0, a]) > 0$.

Note that the inequality (5.3) and condition $\lim_{t \to \infty} A(t) = 0$ imply that $\mu(\{0\}) = 0$ which is essential for constructing positive initial data in $L^1_2(\mathbf{R}^3)$ with the given moment condition.

Now consider initial data of the following type:

$$F(v) = M(v) + \theta_0 h_0(v), \qquad v \in \mathbf{R}^3$$

where $M(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$ is the Maxwellian in $P_2(\mathbf{R}^3; 0, 1)$,

$$h_0(v) = (2\pi)^{-3} \int_{\mathbf{R}^3} e^{-|\xi|^2/2} U_0(\xi)\, e^{i\langle \xi, v\rangle}\, d\xi,$$

$$U_0(\xi) = \int_0^1 |\xi|^{2+s}\, d\mu(s), \qquad \xi \in \mathbf{R}^3,$$

and $\theta_0 > 0$ is a constant.

Lemma 5.1. The function F constructed above has the following properties:

(1) F is an isotropic function: $F(v) = \bar{F}(|v|)$, and $F \in L^1_2(\mathbf{R}^3) \cap H^\infty(\mathbf{R}^3)$, and F is analytic on $\mathbf{R}^3$.

(2)

$$\hat{F}(\xi) = e^{-|\xi|^2/2}(1 + \theta_0 U_0(\xi)), \qquad \int_{\mathbf{R}^3} (1, v, |v|^2)\, F(v)\, dv = (1, 0, 3) \tag{5.4}$$

and for small $\theta_0 > 0$, F is strictly positive on $\mathbf{R}^3$.

(3) There are constants $0 < c := c(\mu, \theta_0)$, $C := C(\mu, \theta_0) < \infty$ which depend only on μ and θ_0, such that

$$c \int_0^1 e^{-st}\, d\mu(s) \leqslant J_F(e^{-t}) \equiv \int_{|v| > e^t} |v|^2\, F(v)\, dv \leqslant C \int_0^1 e^{-st}\, d\mu(s) \qquad t \geqslant 0. \tag{5.5}$$

Proof. First of all we have the following estimate

$$0 \leqslant U_0(\xi) \leqslant \int_0^1 (1 + |\xi|^3)\, d\mu(s) = C(\mu)(1 + |\xi|^3), \quad \xi \in \mathbf{R}^3, \tag{5.6}$$

here and below, the constants $0 < c(*,...)$, $C = C(*,...) < \infty$ depend only on their arguments $*,...$, and $0 < c_0$, $C_0 < \infty$ denote absolute constants. This implies first that the function h_0 and therefore F are well-defined on $\mathbf{R}^3$ and belong to $L^\infty(\mathbf{R}^3) \cap C(\mathbf{R}^3)$. Now we prove a representation of $h_0(v)$:

$$h_0(v) = |v|^{-5}\, (2\pi)^{-3/2} \int_0^1 \frac{1}{w(s)} \int_0^{|v|^2} (|v|^2 - r)^{-s/2}\, r^{(3+s)/2} P(r)\, e^{-r/2}\, dr\, d\mu(s), \quad v \neq 0 \tag{5.7}$$

where

$$w(s) = \int_0^\infty r^{-s/2} e^{-r/2}\, dr, \qquad P(r) = 15 - 10r + r^2. \tag{5.8}$$

Note that this representation implies first that h_0 and therefore F are isotropic functions.

To prove (5.7), we first write $|\xi|^{2+s}$ as follows using change of variable for integral:

$$|\xi|^{2+s} = |\xi|^4 \frac{1}{w(s)} \int_0^\infty r^{-s/2} e^{-r|\xi|^2/2}\, dr, \qquad \xi \in \mathbf{R}^3, \quad 0 \leqslant s \leqslant 1.$$

(Recalling the convention $0 \cdot \infty = 0$, the above equality holds also for $\xi = 0$.) Then by definition of $h_0(v)$ and $U_0(\xi)$, and using Fubini theorem, we have

$$h_0(v) = (2\pi)^{-3/2} \int_0^1 \frac{1}{w(s)} \int_0^\infty r^{-s/2}(1+r)^{-7/2} H(v/\sqrt{1+r}\,)\, dr\, d\mu(s)$$

where

$$H(v) = (2\pi)^{-3/2} \int_{\mathbf{R}^3} |\xi|^4\, e^{-|\xi|^2/2} e^{\mathrm{i}\langle \xi, v\rangle}\, d\xi.$$

Since $e^{-|v|^2/2} = (2\pi)^{-3/2} \int_{\mathbf{R}^3} e^{-|\xi|^2/2} e^{\mathrm{i}\langle \xi, v\rangle}\, d\xi$,

$$H(v) = \Delta^2(e^{-|v|^2/2}) = P(|v|^2)\, e^{-|v|^2/2}$$

where Δ is the Laplacian in $v \in \mathbf{R}^3$ and $P(r)$ is the polynomial in (5.8). From this and changing variables in the integral we obtain

$$\begin{aligned} h_0(v) &= (2\pi)^{-3/2} \int_0^1 \frac{1}{w(s)} \int_0^\infty r^{-s/2}(1+r)^{-7/2} P(|v|^2/(1+r))\, e^{-\frac{1}{2}|v|^2/(1+r)}\, dr\, d\mu(s) \\ &= |v|^{-5}\, (2\pi)^{-3/2} \int_0^1 \frac{1}{w(s)} \int_0^{|v|^2} (|v|^2 - r)^{-s/2}\, r^{(3+s)/2} P(r)\, e^{-r/2}\, dr\, d\mu(s). \end{aligned}$$

This proves (5.7). Next we prove that

$$|h_0(v)| \leqslant C_0 \int_0^1 s\, |v|^{-5-s}\, d\mu(s) + C(\mu)\, |v|^{-7}, \qquad |v| \geqslant 2. \tag{5.9}$$

To do this we consider a decomposition for $|v| \geqslant 2$:

$$\int_0^{|v|^2} (|v|^2 - r)^{-s/2}\, r^{(3+s)/2} P(r)\, e^{-r/2}\, dr = \int_0^{|v|} + \int_{|v|}^{|v|^2}. \tag{5.10}$$

For the first term, we compute using the definition of $P(r)$ and several integrations by parts that for $0 \leqslant s \leqslant 1$ and $|v| \geqslant 2$

$$\begin{aligned}\int_0^{|v|} (|v|^2 - r)^{-s/2}\, r^{(3+s)/2} P(r)\, e^{-r/2}\, dr \\ = (2s+s^2) \int_0^{|v|} (|v|^2-r)^{-s/2}\, r^{(3+s)/2} e^{-r/2}\, dr \\ + ((6-2s)\, |v|^{(5+s)/2} - 2\, |v|^{(7+s)/2})\, e^{-|v|/2} (|v|^2 - |v|)^{-s/2} \\ + \int_0^{|v|} ((s^2-3s)\, r^{(5+s)/2} + s r^{(7+s)/2})(|v|^2-r)^{-(s/2)-1}\, e^{-r/2}\, dr, \qquad |v| \geqslant 2.\end{aligned} \tag{5.11}$$

The computation is lengthly, but what is crucial is the factor $(2s+s^2)$ in the first term after the equal sign. The fact that this integral is multiplied by something that is $\mathcal{O}(s)$ is what we need later on.

Next, by $|v| \geqslant 2$ and $0 \leqslant s \leqslant 1$, we have

$$(|v|^2 - r)^{-s/2} \leqslant 2\, |v|^{-s}, \quad (|v|^2-r)^{-(s/2)-1} \leqslant 4\, |v|^{-2} \qquad \text{for} \quad 0 \leqslant r \leqslant |v|.$$

This together with (5.11) gives

$$\left| \int_0^{|v|} (|v|^2 - r)^{-s/2}\, r^{(3+s)/2} P(r)\, e^{-r/2}\, dr \right| \leqslant C_0 s\, |v|^{-s} + C_0\, |v|^{-2}, \qquad |v| \geqslant 2$$

where $0 < C_0 < \infty$ is an absolute constant.

For the second term in the right-hand side of (5.10), we have for $0 \leqslant s \leqslant 1$ and $|v| \geqslant 2$,

$$\left| \int_{|v|}^{|v|^2} (|v|^2 - r)^{-s/2}\, r^{(3+s)/2} P(r)\, e^{-r/2}\, dr \right| \leqslant C_0\, |v|^{-2}.$$

Thus, for all $|v| \geqslant 2$

$$\left| \int_0^{|v|^2} (|v|^2 - r)^{-s/2}\, r^{(3+s)/2} P(r)\, e^{-r/2}\, dr \right| \leqslant C_0 s\, |v|^{-s} + C_0\, |v|^{-2},$$

from which (5.9) follows, using the fact that on $0 \leqslant s \leqslant 1$, $0 < c_0 \leqslant w(s) \leqslant C_0$.

Next we prove that there is a number $9 < R_0(\mu) < \infty$ such that

$$h_0(v) \geqslant c_0 \int_0^1 s\, |v|^{-5-s}\, d\mu(s), \qquad |v| \geqslant R_0(\mu). \tag{5.12}$$

Since the polynomial $P(r) > 0$ for $r \geqslant 9$, we have by (5.7), (5.10), and (5.11) that for all $|v| \geqslant 9$

$$h_0(v) \geqslant |v|^{-5} (2\pi)^{-3/2} \int_0^1 \frac{1}{w(s)} \int_0^{|v|} (|v|^2 - r)^{-s/2} r^{(3+s)/2} P(r) e^{-r/2} \, dr \, d\mu(s)$$

and

$$\int_0^{|v|} (|v|^2 - r)^{-s/2} r^{(3+s)/2} P(r) e^{-r/2} \, dr \geqslant c_0 s \, |v|^{-s} - C_0 \, |v|^{-2}.$$

Therefore

$$h_0(v) \geqslant |v|^{-7} \left(c_0 \int_0^1 s \, |v|^{2-s} \, d\mu(s) - C(\mu) \right).$$

By $\mu(\{0\}) = 0$ and $\mu([0,1]) = v([0,a]) > 0$ we have $\int_0^1 s \, d\mu(s) > 0$ which implies for some finite number $R_0(\mu) > 9$

$$c_0 \int_0^1 s \, |v|^{2-s} \, d\mu(s) \geqslant c_0 \, |v| \int_0^1 s \, d\mu(s) \geqslant 2C(\mu), \qquad \forall |v| \geqslant R_0(\mu).$$

This gives (5.12).

Now we prove that $F \in L^1_2(\mathbf{R}^3)$. It suffices to prove that $h_0 \in L^1_2(\mathbf{R}^3)$. By (5.9) we have

$$\int_{|v| \geqslant 2} |v|^2 \, |h_0(v)| \, dv \leqslant C_0 \int_0^1 2^{-s} \, d\mu(s) + C(\mu) < \infty.$$

Since $h_0 \in L^\infty(\mathbf{R}^3)$, this implies that h_0 and therefore F belong to $L^1_2(\mathbf{R}^3)$.

Next we prove that for small $\theta_0 > 0$, F is strictly positive on $\mathbf{R}^3$. Let $R_0(\mu)$ be the constant in (5.12) and let $\theta_0 > 0$ be small such that

$$(2\pi)^{-3/2} e^{-(R_0(\mu))^2/2} - \theta_0 \, \|h_0\|_{L^\infty} > 0.$$

Then for any $v \in \mathbf{R}^3$, if $|v| \leqslant R_0(\mu)$, then

$$F(v) = M(v) + \theta_0 h_0(v) \geqslant (2\pi)^{-3/2} e^{-(R_0(\mu))^2/2} - \theta_0 \, \|h_0\|_{L^\infty} > 0;$$

if $|v| \geqslant R_0(\mu)$, then by (5.12) we see that $F(v) \geqslant \theta_0 h_0(v) > 0$. This proves the strict positivity of F.

Now we prove that the function F is analytic on $\mathbf{R}^3$. Equivalently we prove that h_0 is analytic on $\mathbf{R}^3$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be muli-indices with α_i

nonnegative integers, and let $D^\alpha = \partial^{|\alpha|}/\partial\xi_1^{\alpha_1}\,\partial\xi_2^{\alpha_2}\,\partial\xi_3^{\alpha_3}$ be a partial differential operator with order $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. Then from the above estimate and applying the theorem of taking derivative under integral sign we have with $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}$,

$$D^\alpha h_0(v) = (2\pi)^{-3}\,\mathrm{i}^{|\alpha|}\int_{\mathbf{R}^3} e^{-|\xi|^2/2}\left(\int_0^1 |\xi|^{2+s}\,d\mu(s)\right)\xi^\alpha e^{\mathrm{i}\langle\xi,\,v\rangle}\,d\xi.$$

This implies that for any $v, z \in \mathbf{R}^3$ and $n \in \mathbf{N}$

$$\sum_{|\alpha|=n}\left|\frac{D^\alpha h_0(v)}{\alpha!}\,z^\alpha\right| \leqslant C(\mu)\int_{\mathbf{R}^3} e^{-|\xi|^2/2}(1+|\xi|^3)\,\frac{(\sqrt{3}\,|z|\,|\xi|)^n}{n!}\,d\xi.$$

Since

$$\sum_{n=0}^{\infty}\int_{\mathbf{R}^3} e^{-|\xi|^2/2}(1+|\xi|^3)\,\frac{(\sqrt{3}\,|z|\,|\xi|)^n}{n!}\,d\xi = \int_{\mathbf{R}^3} e^{-|\xi|^2/2}(1+|\xi|^3)\,e^{\sqrt{3}\,|z|\,|\xi|}\,d\xi < \infty$$

for all $z \in \mathbf{R}^3$, this implies that the function h_0 is analytic on $\mathbf{R}^3$.

Next we prove (5.4) and the regularity $F \in H^\infty(\mathbf{R}^3)$. It is obvious that the first equality in (5.4) and the bound (5.6) imply $F \in H^\infty(\mathbf{R}^3)$. To prove (5.4), we recall that $F(v) = M(v) + \theta_0 h_0(v)$, and thus it is equivalent to prove that

$$\widehat{h_0}\,(\xi) = e^{-|\xi|^2/2}U_0(\xi) \qquad \text{and} \qquad \int_{\mathbf{R}^3}(1, v, |v|^2)\,h_0(v)\,dv = (0, 0, 0). \tag{5.13}$$

The first equality is obvious because the functions $e^{-|\xi|^2/2}U_0(\xi)$ and $h_0(v)$ are both even, continuous, bounded and integrable on $\mathbf{R}^3$. To prove the second equality in (5.13), we note that $h_0 \in L^1_2(\mathbf{R}^3)$ implies that its Fourier transform $\widehat{h_0} \in C^2(\mathbf{R}^3)$, so we need only to prove that

$$\widehat{h_0}\,(\xi) = o(|\xi|^2) \qquad (|\xi| \to 0).$$

But this is easy: By definition of $U_0(\xi)$ and $\mu(\{0\}) = 0$ we have

$$\frac{\widehat{h_0}\,(\xi)}{|\xi|^2} = e^{-|\xi|^2/2}\int_{(0,\,1]} |\xi|^s\,d\mu(s) \to 0 \qquad (|\xi| \to 0).$$

Finally, we prove the two-sides estimates (5.5). Equivalently we prove that

$$c\int_0^1 R^{-s}\,d\mu(s) \leqslant J_F\left(\frac{1}{R}\right) \equiv \int_{|v|>R} |v|^2\,F(v)\,dv \leqslant C\int_0^1 R^{-s}\,d\mu(s) \qquad \forall R \geqslant 1. \tag{5.14}$$

where the constants $0 < c = c(\mu, \theta_0)$, $C = C(\mu, \theta_0) < \infty$ depend only on μ and θ_0.

To obtain the lower bound in (5.14), we have for any $R \geqslant 1$, if $R \geqslant R_0(\mu)$, then by (5.12) and $\mu(\{0\}) = 0$

$$\int_{|v|>R} |v|^2 F(v)\, dv \geqslant \int_{|v|>R} |v|^2 \left(c_0 \theta_0 \int_{(0,1]} s\, |v|^{-5-s}\, d\mu(s) \right) dv$$
$$= 4\pi c_0 \theta_0 \int_0^1 R^{-s}\, d\mu(s). \tag{5.15}$$

In particular, for $R = R_0(\mu)$ we have

$$\int_{|v|>R_0(\mu)} |v|^2 F(v)\, dv \geqslant 4\pi c_0 \theta_0 \int_0^1 (R_0(\mu))^{-s}\, d\mu(s).$$

This implies that if $1 \leqslant R \leqslant R_0(\mu)$, then, since $F \geqslant 0$,

$$\int_{|v|>R} |v|^2 F(v)\, dv \geqslant \int_{|v|>R_0(\mu)} |v|^2 F(v)\, dv \geqslant 4\pi c_0 \theta_0 \int_0^1 (R_0(\mu))^{-s}\, d\mu(s).$$

Since $1 \leqslant R \leqslant R_0(\mu)$ and $0 \leqslant s \leqslant 1$ we have

$$(R_0(\mu))^{-s} = R^{-s} \left(\frac{R}{R_0(\mu)} \right)^s \geqslant R^{-s} \left(\frac{1}{R_0(\mu)} \right)^s \geqslant R^{-s} \frac{1}{R_0(\mu)} = (R_0(\mu))^{-1} R^{-s}.$$

Thus if $1 \leqslant R \leqslant R_0(\mu)$, then

$$\int_{|v|>R} |v|^2 F(v)\, dv \geqslant \int_{|v|>R_0(\mu)} |v|^2 F(v)\, dv \geqslant 4\pi c_0 \theta_0 \int_0^1 (R_0(\mu))^{-s}\, d\mu(s)$$
$$\geqslant 4\pi c_0 \theta_0 (R_0(\mu))^{-1} \int_0^1 R^{-s}\, d\mu(s).$$

To obtain the upper bound in (5.14), use the estimate (5.9) and obtain that for $R \geqslant 2$

$$\int_{|v|>R} |v|^2 F(v)\, dv = \int_{|v|>R} M(v)\, dv + \theta_0 \int_{|v|>R} |v|^2 h_0(v)\, dv$$
$$\leqslant C(\mu)\, R^{-1} + \theta_0 C_0 \int_{(0,1]} s \left(\int_{|v|>R} |v|^{-3-s}\, dv \right) d\mu(s)$$
$$\leqslant \left(\frac{C(\mu)}{\mu([0,1])} + \theta_0 C_0 \right) \int_0^1 R^{-s}\, d\mu(s)$$

and for $1 \leqslant R \leqslant 2, \int_{|v|>R} |v|^2 F(v)\, dv \leqslant 3 \leqslant 3(\int_0^1 2^{-s}\, d\mu(s))^{-1} \int_0^1 R^{-s}\, d\mu(s)$. This proves the estimate (5.14) and therefore (5.5) holds true. ∎

5.2. Linear and Nonlinear Solutions: A Comparison Property

Let U_0, h_0 and $F = M + \theta_0 h_0$ be the functions constructed in the Part 1 with a small constant $\theta_0 > 0$. Since $F \in L^1_2(\mathbf{R}^3)$ is positive, there is a unique solution $f_t \geqslant 0$ of the Boltzmann equation (1.1) in $C^1([0, \infty); L^1_2(\mathbf{R}^3))$ with the initial datum $f_t|_{t=0} = F$ and f_t conserves the mass, momentum and energy, i.e., $\int_{\mathbf{R}^3} (1, v, |v|^2)\, f_t(v)\, dv = (1, 0, 3)$ for all $t \geqslant 0$. Note that since the initial datum $F(v) = \bar{F}(|v|)$ is isotropic, it is easily seen (using the uniqueness) that the solution $f_t(v)$ and therefore its Fourier transform $\hat{f}_t(\xi)$ are also isotropic, i.e.,

$$f_t(v) = \bar{f}(|v|, t), \qquad \hat{f}_t(\xi) = f^*(|\xi|, t), \qquad v, \xi \in \mathbf{R}^3, \quad t \geqslant 0.$$

Let

$$U(\xi, t) = e^{|\xi|^2/2} \hat{f}_t(\xi) - 1 = e^{|\xi|^2/2} f^*(|\xi|, t) - 1 \qquad \xi \in \mathbf{R}^3, \quad t \geqslant 0 \tag{5.16}$$

or equivalently

$$\hat{f}_t(\xi) = f^*(|\xi|, t) = e^{-|\xi|^2/2}(1 + U(\xi, t)), \qquad \xi \in \mathbf{R}^3, \quad t \geqslant 0.$$

It is easily checked that $U(\xi, t)$ is real and continuous on $\mathbf{R}^3 \times [0, \infty)$ and satisfies the following Boltzmann equation of the Fourier transform version:

$$\frac{\partial}{\partial t} U(\xi, t) = \int_0^\pi \bar{B}(\theta)[-U(\xi, t) + U(\cos(\theta/2)\, \xi, t) + U(\sin(\theta/2)\, \xi, t)]\, d\theta$$
$$+ \int_0^\pi \bar{B}(\theta)[U(\cos(\theta/2)\, \xi, t)\, U(\sin(\theta/2)\, \xi, t)]\, d\theta \tag{5.17}$$

for $\xi \in \mathbf{R}^3, t \geqslant 0$ with the initial datum $U|_{t=0} = \theta_0 U_0$. Here $\bar{B}(\theta) = 2\pi B(\cos(\theta)) \sin(\theta)$.

Now we turn to the linear equation: Let $Y_0 \in C(\mathbf{R}^3)$ and $Y_0 \geqslant 0$ on $\mathbf{R}^3$. Let $Y(\xi, t)$ be a solution of the following linear equation (i.e., neglect the nonlinear term in Eq. (5.17))

$$\frac{\partial}{\partial t} Y(\xi, t) = \int_0^\pi \bar{B}(\theta)[-Y(\xi, t) + Y(\cos(\theta/2)\, \xi, t) + Y(\sin(\theta/2)\, \xi, t)]\, d\theta \tag{5.18}$$

for all $(\xi, t) \in \mathbf{R}^3 \times [0, \infty)$, and satisfy the initial condition $Y|_{t=0} = Y_0$. Existence and uniqueness of the continuous solution Y is easily proven and $Y(\xi, \cdot) \in C^1[0, \infty)$ for each fixed $\xi \in \mathbf{R}^3$. Moreover, Y is nonnegative on $\mathbf{R}^3 \times [0, \infty)$. Here we only prove the nonnegativeness. It suffices to show that for any $R > 0$, Y is nonnegative on the set $\{(\xi, t) \in \mathbf{R}^3 \times [0, \infty) \mid |\xi| \leqslant R\}$. Note that the function $y \mapsto (-y)^+ := \max\{-y, 0\}$ is a Lipschitz-function on $\mathbf{R}$. By $Y_0 \geqslant 0$, we have $(-Y_0(\xi))^+ = 0$ and

$$
\begin{aligned}
(-Y(\xi, t))^+ &= (-Y_0(\xi))^+ + \int_0^t \left(-\frac{\partial}{\partial \tau} Y(\xi, \tau)\right) 1_{\{Y(\xi, \tau) < 0\}} \, d\tau \\
&= \int_0^t \int_0^\pi \bar{B}(\theta)((Y(\xi, \tau) \, 1_{\{Y(\xi, \tau) < 0\}} - Y(\cos(\theta/2) \, \xi, \tau) \, 1_{\{Y(\xi, \tau) < 0\}} \\
&\quad - Y(\sin(\theta/2) \, \xi, \tau) \, 1_{\{Y(\xi, \tau) < 0\}}) \, d\theta \, d\tau \\
&\leqslant \int_0^t \int_0^\pi \bar{B}(\theta)((-Y(\cos(\theta/2) \, \xi, \tau))^+ + (-Y((\sin(\theta/2) \, \xi, \tau))^+) \, d\theta \, d\tau \\
&\leqslant 2 \int_0^t h_R(\tau) \, d\tau, \quad |\xi| \leqslant R, \quad t \in [0, \infty).
\end{aligned}
$$

Here, $h_R(t) = \sup_{|\xi| \leqslant R} (-Y(\xi, t))^+$. Therefore, we obtain

$$
h_R(t) \leqslant 2 \int_0^t h_R(\tau) \, d\tau, \qquad t \in [0, \infty).
$$

Since $Y(\xi, t)$ is continuous on $\mathbf{R}^3 \times [0, \infty)$, the function $h_R(\cdot)$ is locally bounded on $[0, \infty)$. Thus the Gronwall lemma can be used which implies that $h_R(t) \equiv 0$ on $[0, \infty)$. This means that $Y(\xi, t) \geqslant 0$ on $\{(\xi, t) \in \mathbf{R}^3 \times [0, \infty) \mid |\xi| \leqslant R\}$. Since $R > 0$ is arbitrary, it follows that $Y(\xi, t) \geqslant 0$ for all $(\xi, t) \in \mathbf{R}^3 \times [0, \infty)$.

Using the same argument (Gronwall lemma, local boundedness, etc.), it can be proven that the continuous solution of the non-linear Eq. (5.17) is also unique and is nonnegative provided that its initial datum is continuous and nonnegative. Generally, we have the following comparison property:

Lemma 5.2. Let $V_0(\xi)$, $Y_0(\xi)$ be continuous on $\mathbf{R}^3$ and satisfy

$$
V_0(\xi) \geqslant Y_0(\xi) \geqslant 0, \qquad \forall \xi \in \mathbf{R}^3.
$$

Let $V(\xi, t)$, $Y(\xi, t)$ be continuous solutions of the non-linear equation (5.17) and the linear equation (5.18) on $\mathbf{R}^3 \times [0, \infty)$ respectively, and satisfy $V|_{t=0} = V_0$, $Y|_{t=0} = Y_0$. Then

$$
V(\xi, t) \geqslant Y(\xi, t) \geqslant 0 \qquad \forall (\xi, t) \in \mathbf{R}^3 \times [0, \infty).
$$

Proof. We have proved that the solution $Y(\xi, t)$ is nonnegative on $\mathbf{R}^3 \times [0, \infty)$ since Y_0 is nonnegative. Therefore, we need only prove that $V(\xi, t) \geqslant Y(\xi, t)$. Let

$$W(\xi, t) = V(\xi, t) - Y(\xi, t), \qquad W_0(\xi) = V_0(\xi) - Y_0(\xi).$$

For any $0 < R < \infty$ and any $0 < T < \infty$, let

$$H_R(t) = \sup_{|\xi| \leqslant R} (-W(\xi, t))^+, \qquad C_{R,T} = \sup_{|\xi| \leqslant R,\, 0 \leqslant t \leqslant T} \max\{|W(\xi, t)|, Y(\xi, t)\}.$$

Then $C_{R,T} < \infty$. In our derivation below we will use the following simple but useful inequalities: For any $w_1, w_2, w \in \mathbf{R}$, $y \geqslant 0$ and $0 \leqslant \chi \leqslant 1$, we have

$$-w_1 w_2 \chi \leqslant (-w_1)^+ |w_2| + (-w_2)^+ |w_1|, \qquad -wy\chi \leqslant (-w)^+ y. \tag{5.19}$$

By (5.17), (5.18), and using the property (5.19) and notice that $W_0 \geqslant 0$, $Y \geqslant 0$ we have for any $(\xi, t) \in \mathbf{R}^3 \times [0, \infty)$ satisfying $|\xi| \leqslant R$, $t \leqslant T$

$$
\begin{aligned}
&(-W(\xi, t))^+ \\
&\quad = (-W_0(\xi))^+ + \int_0^t \left(-\frac{\partial}{\partial \tau} W(\xi, \tau) \right) 1_{\{W(\xi, \tau) < 0\}}\, d\tau \\
&\quad = \int_0^t \left\{ \int_0^\pi \bar{B}(\theta)\, [W(\xi, \tau)\, 1_{\{W(\xi, \tau) < 0\}} - W(\cos(\theta/2)\, \xi, \tau)\, 1_{\{W(\xi, \tau) < 0\}} \right. \\
&\qquad - W(\sin(\theta/2)\, \xi, \tau)\, 1_{\{W(\xi, \tau) < 0\}} \\
&\qquad - W(\cos(\theta/2)\, \xi, \tau)\, W(\sin(\theta/2)\, \xi, \tau)\, 1_{\{W(\xi, \tau) < 0\}} \\
&\qquad - W(\cos(\theta/2)\, \xi, \tau)\, Y(\sin(\theta/2)\, \xi, \tau)\, 1_{\{W(\xi, \tau) < 0\}} \\
&\qquad - Y(\cos(\theta/2)\, \xi, \tau)\, W(\sin(\theta/2)\, \xi, \tau)\, 1_{\{W(\xi, \tau) < 0\}} \\
&\qquad \left. - Y(\cos(\theta/2)\, \xi, \tau)\, Y(\sin(\theta/2)\, \xi, \tau)\, 1_{\{W(\xi, \tau) < 0\}}]\, d\theta \right\} d\tau \\
&\quad \leqslant \int_0^t \left\{ \int_0^\pi \bar{B}(\theta) [(-W(\cos(\theta/2)\, \xi, \tau))^+ + (-W(\sin(\theta/2)\, \xi, \tau))^+ \right. \\
&\qquad + (-W(\cos(\theta/2)\, \xi, \tau))^+\, |W(\sin(\theta/2)\, \xi, \tau)| \\
&\qquad + (-W(\sin(\theta/2)\, \xi, \tau))^+\, |W(\cos(\theta/2)\, \xi, \tau)| \\
&\qquad + (-W(\cos(\theta/2)\, \xi, \tau))^+\, Y(\sin(\theta/2)\, \xi, \tau) \\
&\qquad \left. + (-W(\sin(\theta/2)\, \xi, \tau))^+\, Y(\cos(\theta/2)\, \xi, \tau)]\, d\theta \right\} d\tau \\
&\quad \leqslant \int_0^t \left\{ \int_0^\pi \bar{B}(\theta) [2H_R(\tau) + 4C_{R,T} H_R(\tau)]\, d\theta \right\} d\tau = (2 + 4C_{R,T}) \int_0^t H_R(\tau)\, d\tau.
\end{aligned}
$$

Thus by the definition of $H_R(t)$, we obtain

$$H_R(t) \leqslant (2+4C_{R,T}) \int_0^t H_R(\tau)\, d\tau \qquad \forall t \in [0, T].$$

By Gronwall's lemma, this implies that $H_R(t) \equiv 0$ on $[0, T]$. Equivalently, $W(\xi, t) \geqslant 0 \forall\, |\xi| \leqslant R$, $\forall t \in [0, T]$. Letting $R \to \infty, T \to \infty$ gives the global comparison:

$$V(\xi, t) - Y(\xi, t) = W(\xi, t) \geqslant 0 \qquad \forall (\xi, t) \in \mathbf{R}^3 \times [0, \infty). \quad \blacksquare$$

Remark. As one can see, in this comparison property, the nonnegativity of the initial data and the signs in the integrands in (5.17) and (5.18), are essential. These conditions come from the physical nature of the Boltzmann equation, and in particular, its irreversibility.

5.3. Lower Bounds and the Scale-Equivalence

Now we will use the comparison property to estimate the lower bounds of the rate of convergence to equilibrium. Let μ be the positive Borel measure on $[0, 1]$ obtained above. For any $s \geqslant 0$, let

$$\lambda(s) = \int_0^\pi \bar{B}(\theta)(1 - (\cos(\theta/2))^{2+s} - (\sin(\theta/2))^{2+s})\, d\theta, \tag{5.20}$$

$$Y(\xi, t) = \theta_0 \int_0^1 |\xi|^{2+s} e^{-\lambda(s)\, t}\, d\mu(s), \qquad (\xi, t) \in \mathbf{R}^3 \times [0, \infty). \tag{5.21}$$

Here and below the constant $\theta_0 = \theta_0(\mu) > 0$ is fixed and such that the initial datum $F(v)$ is strictly positive on $\mathbf{R}^3$. Let $f_t(v)$ be the unique solution of the Boltzmann equation in $C^1([0, \infty); L^1_2(\mathbf{R}^3))$ with $f_t|_{t=0} = F$, and let $M = M_F$ be the corresponding Maxwellian.

Lemma 5.3. Using the above notation,

$$\hat{f}_t(\xi) - \hat{M}(\xi) \geqslant e^{-|\xi|^2/2} Y(\xi, t) \qquad \forall \xi \in \mathbf{R}^3, \quad \forall t \geqslant 0, \tag{5.22}$$

$$-\lambda(s) \geqslant -s/2, \qquad \forall s \in [0, 1] \tag{5.23}$$

and there is a constant $c > 0$ such that

$$[\hat{f}_t(\xi) - \hat{M}(\xi)]|_{|\xi|=1} \geqslant cA(t), \qquad \forall t \geqslant 0. \tag{5.24}$$

Proof. As mentioned above, the function $U(\xi, t)$ defined in (5.16) is a continuous solution of the nonlinear Eq. (5.17) with the initial datum $U|_{t=0} = \theta_0 U_0$. By definition of $\lambda(s)$, it is easily checked that the function $Y(\xi, t)$ given in (5.21) is a continuous solution of the linear Eq. (5.18) with $Y|_{t=0} = \theta_0 U_0$. Since $U|_{t=0} = Y|_{t=0}$, it follows from the comparison property that $U(\xi, t) \geqslant Y(\xi, t)$ for all $(\xi, t) \in \mathbf{R}^3 \times [0, \infty)$. Since $\hat{f}_t(\xi) - \hat{M}(\xi) = e^{-|\xi|^2/2} U(\xi, t)$, this proves (5.22). Next we estimate $\lambda(s)$. Since $s \geqslant 0$, we have by convexity

$$(\cos(\theta/2))^{2+s} + (\sin(\theta/2))^{2+s} \geqslant 2^{-s/2} \geqslant 1 - s/2.$$

By (5.20), this gives (5.23):

$$-\lambda(s) = \int_0^\pi \bar{B}(\theta)(-1 + (\cos(\theta))^{2+s} + (\sin(\theta/2))^{2+s})\, d\theta \geqslant -s/2.$$

Now for $|\xi| = 1$, $Y(\xi, t) = \theta_0 \int_0^1 e^{-\lambda(s)\, t}\, d\mu(s) \geqslant \theta_0 \int_0^1 e^{-st/2}\, d\mu(s)$ for all $t \geqslant 0$. This estimate together with (5.22), (5.3) and $1/2 \leqslant a$ gives (5.24): For all $t \geqslant 0$

$$\begin{aligned}[\hat{f}_t(\xi) - \hat{M}(\xi)]|_{|\xi|=1} &\geqslant e^{-1/2} Y(\xi, t)|_{|\xi|=1} \geqslant e^{-1/2} \theta_0 \int_0^1 e^{-st/2}\, d\mu(s) \\ &\geqslant cA(t/(2a)) \geqslant cA(t). \quad \blacksquare\end{aligned}$$

Proof of Theorem 4. From (5.24) we obtain the lower bounds (1.29) and (1.30) in the Theorem 4:

$$\begin{aligned}&\|f_t - M\|_{L^1} \geqslant \|\hat{f}_t - \hat{M}\|_{L^\infty} \geqslant [\hat{f}_t(\xi) - \hat{M}(\xi)]|_{|\xi|=1} \geqslant cA(t), \qquad t \geqslant 0, \\ &\|f_t - M\|_{L^1_2} \geqslant |\!|\!| f_t - M |\!|\!|_0 \geqslant [\hat{f}_t(\xi) - \hat{M}(\xi)]|_{|\xi|=1} \geqslant cA(t), \qquad t \geqslant 0.\end{aligned}$$

Here we have used the inequality (2.11) because $\int_{\mathbf{R}^3} (1, v, |v|^2)\, f_t(v)\, dv = (1, 0, 3)$ for all $t \geqslant 0$.

Next we prove that if the kernel $B(\cdot)$ satisfies the Hölder condition (1.25), then the t-functions $\|f_t - M\|_{L^1_2}$, $\|f_t - M\|_{L^1}$, $\Phi_{B,F}(e^{-t})$, $J_F(e^{-t})$, $A(t)$, $\|\hat{f}_t - \hat{M}\|_{L^\infty}$ and $|\!|\!| f_t - M |\!|\!|_0$ are scale-equivalent each other. We will prove this by showing that the following relation of a closed chain holds:

$$\begin{aligned}&|\!|\!| f_t - M |\!|\!|_0 \preccurlyeq \|f_t - M\|_{L^1_2} \preccurlyeq \Phi_{B,F}(e^{-t}) \preccurlyeq J_F(e^{-t}) \preccurlyeq A(t) \\ &\qquad \preccurlyeq \|\hat{f}_t - \hat{M}\|_{L^\infty} \preccurlyeq \|f_t - M\|_{L^1} \preccurlyeq A(t) \preccurlyeq |\!|\!| f_t - M |\!|\!|_0 \quad \text{on } t \in [0, \infty).\end{aligned}$$

The first and the second "$\preccurlyeq$" have been proved by the lower bounds (1.30) and by Theorem 2 respectively. And because of (1.30) and (1.29), the last "$\preccurlyeq$" is obvious and $cA(t) \leqslant \|\hat{f}_t - \hat{M}\|_{L^\infty} \leqslant \|f_t - M\|_{L^1} \leqslant \|f_t - M\|_{L^1_2}$. Thus to complete the proof, we need only to prove that $\Phi_{B,F}(e^{-t}) \preccurlyeq J_F(e^{-t}) \preccurlyeq A(t)$ on $t \in [0, \infty)$.

By definition of $\Phi_{B,F}$ and the Hölder condition $\Omega_B(r) \leqslant C_B r^\alpha$ we have

$$\begin{aligned}\Phi_{B,F}(e^{-t}) &= C\Omega_B^*((e^{-t} + J_F(e^{-t}))^{1/60}) \\ &\leqslant C[\Omega_B((e^{-t} + J_F(e^{-t}))^{1/60})]^{6/7} \leqslant C(e^{-t} + J_F(e^{-t}))^{\alpha/70}.\end{aligned}$$

On the other hand, using (5.5) we have

$$J_F(e^{-t}) \geqslant c \int_0^1 e^{-st}\, d\mu(s) \geqslant c\mu([0, 1])\, e^{-t}, \qquad t \geqslant 0.$$

Since $0 < c\mu([0, 1]) < \infty$, this implies that $e^{-t} \leqslant C J_F(e^{-t})$ and so

$$\Phi_{B,F}(e^{-t}) \leqslant C(J_F(e^{-t}))^{\alpha/70}, \qquad t \geqslant 0.$$

Thus $\Phi_{B,F}(e^{-t}) \preccurlyeq J_F(e^{-t})$ on $t \in [0, \infty)$. Again, using (5.5) and (5.3) we have

$$J_F(e^{-t}) \leqslant C \int_0^1 e^{-st}\, d\mu(s) \leqslant CA(t/a), \qquad t \geqslant 0.$$

This proves $J_F(e^{-t}) \preccurlyeq A(t)$ on $t \in [0, \infty)$. ∎

We conclude this section with the following lemma, which has been mentioned in the introduction.

Lemma 5.4. If $A_0(t)$ is any completely monotone function on $[0, \infty)$, then so is $A_1(t) = A_0(\log(1+t))$.

Proof. It is obvious that A_1 is positive, $A_1 \in C^\infty(0, \infty) \cap C[0, \infty)$,

$$A_1(0) = A_0(0) = 1 \qquad \text{and} \qquad \lim_{t \to \infty} A_1(t) = \lim_{x \to \infty} A_0(x) = 0.$$

Thus by definition of completely monotone, we need only to prove that

$$(-1)^n \frac{d^n}{dt^n} A_1(t) \geqslant 0 \qquad \forall n = 1, 2, \ldots, \qquad t > 0.$$

For $n=1, 2$ we have

$$-\frac{d}{dt}A_1(t)=-\left(\frac{d}{dx}A_0\right)(\log(1+t))\,(1+t)^{-1}\geqslant 0, \qquad t>0$$

$$\frac{d^2}{dt^2}A_1(t)=(1+t)^{-2}\left\{\left(\frac{d^2}{dx^2}A_0\right)(\log(1+t))-\left(\frac{d}{dx}A_0\right)(\log(1+t))\right\}\geqslant 0, \quad t>0.$$

Suppose that for some integer $n\geqslant 2$

$$(-1)^n\frac{d^n}{dt^n}A_1(t)=(1+t)^{-n}\sum_{k=1}^{n}a_k^{(n)}(-1)^k\left(\frac{d^k}{dx^k}A_0\right)(\log(1+t)), \qquad t>0 \tag{5.25}$$

with $a_k^{(n)}>0, \quad k=1,..., n$. Then we compute

$$\begin{aligned}(-1)^{n+1}&\frac{d^{n+1}}{dt^{n+1}}A_1(t)\\ &=(1+t)^{-n-1}\sum_{k=1}^{n+1}a_k^{(n+1)}(-1)^k\left(\frac{d^k}{dx^k}A_0\right)(\log(1+t)), \qquad t>0\end{aligned}$$

where

$$a_1^{(n+1)}=na_1^{(n)}>0,\ a_k^{(n+1)}=na_k^{(n)}+a_{k-1}^{(n)}>0,\ k=2, 3,..., n;\ a_{n+1}^{(n+1)}=a_n^{(n)}>0.$$

This proves that the equality (5.25) with all $a_k^{(n)}>0$ holds true for all $n=1, 2, 3,...$. Thus $A_1(t)$ is completely monotone on $[0, \infty)$. ▮

6. GLOBAL STABILITY

In this section we prove Theorem 3 and its Corollary. Before proving the theorem, let us explain why assume that F is not a Dirac measure. Suppose that $F\in\mathscr{B}_2(\mathbf{R}^3)$ is a Dirac measure. Without loss generality we assume that $F\in P_2(\mathbf{R}^3)$ and F concentrates on $0\in\mathbf{R}^3$, i.e., $\int_{\mathbf{R}^3}\phi(v)\,dF(v)=\phi(0)\ \forall\phi\in C(\mathbf{R}^3)$. Then we have

$$\forall G\in P_2(\mathbf{R}^3)\text{ satisfying } G\neq F\ \Rightarrow \lim_{t\to\infty}\|f_t-g_t\|_{\mathscr{B}_2}\geqslant\lim_{t\to\infty}\|f_t-g_t\|_{\mathscr{B}_0}=2. \tag{6.1}$$

Here f_t and g_t are solutions of the Boltzmann equation (1.19) with $f_t|_{t=0}=F$ and $g_t|_{t=0}=G$. Note that since F is a Dirac measure, it is an equilibrium, i.e., $f_t\equiv F$.

(6.1) shows that even if the initial distance $\|F-G\|_{\mathscr{B}_0}$ can be small, for instance $G=(1-\varepsilon)\,F+\varepsilon M$ with $0<\varepsilon\ll 1$, the distance $\|f_t-g_t\|_{\mathscr{B}_0}$ of corresponding solutions can not be small uniformly unless $G=F$.

The proof of (6.1) is as follows: We first prove the following property:

$$H\in P_2(\mathbf{R}^3) \quad \text{and} \quad H(\{0\})=0 \Rightarrow \|F-H\|_{\mathscr{B}_0}=2. \tag{6.2}$$

Since $\|F-H\|_{\mathscr{B}_0}\leqslant 2$, we need only to prove that $\|F-H\|_{\mathscr{B}_2}\geqslant 2$. Consider the following test functions $\phi_N(v)$: For each integer $N\geqslant 2$, define a continuous function $\bar{\phi}_N(r)$ on $[0,\infty)$ by $\bar{\phi}_N(r)=1$ for $0\leqslant r\leqslant 1/N$; $\bar{\phi}_N(r)=-1$ for $2/N\leqslant r\leqslant N$; $\bar{\phi}_N(r)=0$ for $r\geqslant 2N$, and $|\bar{\phi}_N(r)|\leqslant 1$ for all $r\geqslant 0$. Let $\phi_N(v)=\bar{\phi}_N(|v|)$. Then we have, by $\|\phi_N\|_{L^\infty}=1$ and $H(\{0\})=0$

$$\begin{aligned}\|F-H\|_{\mathscr{B}_0} &\geqslant \left|\int_{\mathbf{R}^3}\phi_N(v)\,dF(v)-\int_{\mathbf{R}^3}\phi_N(v)\,dH(v)\right| \\ &\geqslant 2-2\int_{0<|v|<2/N} dH(v)-\frac{2}{N^2}\int_{\mathbf{R}^3}|v|^2\,dH(v)\to 2 \qquad (N\to\infty).\end{aligned}$$

Now we prove (6.1). If G is not a Dirac measure, then $T_G=\frac{1}{3}\int_{\mathbf{R}^3}|v-u_G|^2\,dG(v)>0$ where u_G is the mean velocity. By Theorem 2 we have $\lim_{t\to\infty}\|g_t-M_G\|_{\mathscr{B}_0}=0$. Thus, recalling that $f_t\equiv F$ is an equilibrium and using the property (6.2) we obtain

$$\lim_{t\to\infty}\|f_t-g_t\|_{\mathscr{B}_2}\geqslant \lim_{t\to\infty}\|f_t-g_t\|_{\mathscr{B}_0}=\|F-M_G\|_{\mathscr{B}_0}=2.$$

If G is a Dirac measure, i.e., G concentrates on u_G, then $u_G\neq 0$ since $G\neq F$. This implies that $G(\{0\})=0$. Since $g_t\equiv G$ is an equilibrium, it follows from (6.2) that

$$\|f_t-g_t\|_{\mathscr{B}_2}\geqslant \|f_t-g_t\|_{\mathscr{B}_0}=\|F-G\|_{\mathscr{B}_0}=2, \qquad t\geqslant 0.$$

This proves (6.1).

Proof of Theorem 3. Let F be given in the theorem and let $0\leqslant G\in\mathscr{B}_2(\mathbf{R}^3)$. Let f_t, g_t be distributional solutions of the Boltzmann equation (1.19) in $C([0,\infty);\mathscr{B}_2(\mathbf{R}^3,\|\cdot\|_{\mathscr{B}_2}))$ with $f_t|_{t=0}=F$, $g_t|_{t=0}=G$. For a technical reason we define

$$D_F=\min\left\{1,\frac{1}{2}\|F\|_{\mathscr{B}_0},\frac{3(\|F\|_{\mathscr{B}_0})^3\,T_F}{16(\|F\|_{\mathscr{B}_2})^2},\frac{\|F\|_{\mathscr{B}_0}}{(12+8\,|u_F|)\,\|F\|_{\mathscr{B}_2}}\int_{\mathbf{R}^3}|v-u_F|\,dF(v)\right\}.$$

By assumption on F, it is obvious that $D_F > 0$. If $\|F-G\|_{\mathscr{B}_2} \geqslant D_F$, then by conservation law (1.18)

$$\|f_t - g_t\|_{\mathscr{B}_2} \leqslant \|F\|_{\mathscr{B}_2} + \|G\|_{\mathscr{B}_2} \leqslant C_F \,\|F-G\|_{\mathscr{B}_2}.$$

Here and below the constant C_F depends only on $|u_F|$, T_F, $\|F\|_{\mathscr{B}_0}$, and $\int_{\mathbf{R}^3} |v-u_F|\, dF(v)$, i.e.,

$$C_F = C\left(|u_F|, T_F, \|F\|_{\mathscr{B}_0}, \int_{\mathbf{R}^3} |v-u_F|\, dF(v)\right)$$

and where the function $(y_1, y_2, y_3, y_4) \mapsto C(y_1, y_2, y_3, y_4)$ is continuous on $[0, \infty) \times (0, \infty)^3$ and has an explicit representation in terms of its arguments y_i. Note that $\|F\|_{\mathscr{B}_2}$ is a function of $(|u_F|, T_F, \|F\|_{\mathscr{B}_0})$. In fact $\|F\|_{\mathscr{B}_2} = (1+|u_F|^2+3T_F)\,\|F\|_{\mathscr{B}_0}$. Also, if $\|F-G\|_{\mathscr{B}_2} = 0$, then the uniqueness implies that the estimate (1.26) in the theorem holds true. Therefore in the following we assume that

$$0 < \|F-G\|_{\mathscr{B}_2} < D_F. \tag{6.3}$$

Note that this implies first that G is not a Dirac measure! (See below).

Before estimating $\|f_t - g_t\|_{\mathscr{B}_2}$, we collect some elementary estimates which are easily checked using the condition (6.3) and definition of D_F.

$$\frac{1}{2}\,\|F\|_{\mathscr{B}_0} \leqslant \|G\|_{\mathscr{B}_0} \leqslant \|G\|_{\mathscr{B}_2} \leqslant \frac{3}{2}\,\|F\|_{\mathscr{B}_2},$$

$$\left\| \frac{F}{\|F\|_{\mathscr{B}_0}} - \frac{G}{\|G\|_{\mathscr{B}_0}} \right\|_{\mathscr{B}_2} \leqslant \frac{4\,\|F\|_{\mathscr{B}_2}}{(\|F\|_{\mathscr{B}_0})^2}\,\|F-G\|_{\mathscr{B}_2}, \tag{6.4}$$

$$|u_F - u_G| \leqslant \frac{2\,\|F\|_{\mathscr{B}_2}}{(\|F\|_{\mathscr{B}_0})^2}\,\|F-G\|_{\mathscr{B}_2}, \qquad |u_G| \leqslant \frac{3}{2}\,\frac{\|F\|_{\mathscr{B}_2}}{\|F\|_{\mathscr{B}_0}}, \tag{6.5}$$

and

$$\big|\,|u_F|^2 - |u_G|^2\big| \leqslant \frac{4(\|F\|_{\mathscr{B}_2})^2}{(\|F\|_{\mathscr{B}_0})^3}\,\|F-G\|_{\mathscr{B}_2}$$

which together with

$$3T_F = \int_{\mathbf{R}^3} |v|^2 \frac{dF(v)}{\|F\|_{\mathscr{B}_0}} - |u_F|^2$$

gives

$$|T_F - T_G| \leqslant \frac{8(\|F\|_{\mathscr{B}_2})^2}{3(\|F\|_{\mathscr{B}_0})^3} \|F-G\|_{\mathscr{B}_2}. \tag{6.6}$$

And finally, the condition $\|F-G\|_{\mathscr{B}_2} < D_F$ implies that

$$0 < \tfrac{1}{2} T_F \leqslant T_G \leqslant \tfrac{3}{2} T_F. \tag{6.7}$$

Now by distributional version (1.19) of the Boltzmann equation we have

$$\|f_t - g_t\|_{\mathscr{B}_2} \leqslant \|F-G\|_{\mathscr{B}_2} + \int_0^t \|f_\tau \circ f_\tau - g_\tau \circ g_\tau\|_{\mathscr{B}_2}\, d\tau$$
$$+ \int_0^t \| \|F\|_{\mathscr{B}_0} f_\tau - \|G\|_{\mathscr{B}_0} g_\tau \|_{\mathscr{B}_2}\, d\tau.$$

By the conservation law (1.18) we have

$$\|f_\tau \circ f_\tau - g_\tau \circ g_\tau\|_{\mathscr{B}_2} \leqslant (\|F\|_{\mathscr{B}_2} + \|G\|_{\mathscr{B}_2})\, \|f_\tau - g_\tau\|_{\mathscr{B}_2},$$
$$\| \|F\|_{\mathscr{B}_0} f_\tau - \|G\|_{\mathscr{B}_0} g_\tau \|_{\mathscr{B}_2} \leqslant (\|F\|_{\mathscr{B}_2} + \|G\|_{\mathscr{B}_2})\, \|f_\tau - g_\tau\|_{\mathscr{B}_2}.$$

Since $\|G\|_{\mathscr{B}_2} \leqslant \frac{3}{2} \|F\|_{\mathscr{B}_2}$, this gives

$$\|f_t - g_t\|_{\mathscr{B}_2} \leqslant \|F-G\|_{\mathscr{B}_2} + 5\, \|F\|_{\mathscr{B}_2} \int_0^t \|f_\tau - g_\tau\|_{\mathscr{B}_2}\, d\tau, \qquad t \in [0, \infty)$$

so by $f_t, g_t \in C([0, \infty); \mathscr{B}_2(\mathbf{R}^3, \|\cdot\|_{\mathscr{B}_2}))$ and Gronwall lemma we obtain

$$\|f_t - g_t\|_{\mathscr{B}_2} \leqslant \|F-G\|_{\mathscr{B}_2}\, e^{5\, \|F\|_{\mathscr{B}_2} t}, \qquad t \in [0, \infty). \tag{6.8}$$

On the hand, we have

$$\|f_t - g_t\|_{\mathscr{B}_2} \leqslant \|f_t - M_F\|_{\mathscr{B}_2} + \|g_t - M_G\|_{\mathscr{B}_2} + \|M_F - M_G\|_{\mathscr{B}_2}. \tag{6.9}$$

Here M_F, M_G are the corresponding Maxwellians. Since the distribution $t \mapsto \frac{1}{a} f(\frac{t}{a})$ is a solution to the Boltzmann equation with initial datum $\frac{1}{a} F \in P_2(\mathbf{R}^3)$ where $a = \|F\|_{\mathscr{B}_0} > 0$, by Theorem 2 we have

$$\|f_t - M_F\|_{\mathscr{B}_2} = a \left\| \frac{1}{a} f\left(\frac{at}{a}\right) - \frac{1}{a} M_F \right\|_{\mathscr{B}_2} \leqslant a \Phi_{B, \frac{1}{a} F}(e^{-\beta a t}), \qquad t \in [0, \infty) \tag{6.10}$$

and similarly with $\tilde{a} = \|G\|_{\mathscr{B}_0}$

$$\|g_t - M_G\|_{\mathscr{B}_2} \leqslant \tilde{a}\Phi_{B,\frac{1}{\tilde{a}}G}(e^{-\beta\tilde{a}t}), \quad t \in [0, \infty). \tag{6.11}$$

Now we estimate $\tilde{a}\Phi_{B,\frac{1}{\tilde{a}}G}(e^{-\beta\tilde{a}t})$. By monotonicity of $\Omega_B^*(r)$ and $\|G\|_{\mathscr{B}_0} \geqslant \frac{1}{2}\|F\|_{\mathscr{B}_0}$ we have, with $a = \|F\|_{\mathscr{B}_0}$,

$$\begin{aligned}
\tilde{a}\Phi_{B,\frac{1}{\tilde{a}}G}(e^{-\beta\tilde{a}t}) &\leqslant \frac{3}{2}\|F\|_{\mathscr{B}_2}\, \Phi_{B,\frac{1}{\tilde{a}}G}(e^{-\frac{1}{2}\beta a t}) \\
&= \frac{3}{2}\|F\|_{\mathscr{B}_2} C_B (1+|u_G|)^3\,(1+T_G)^2 \left(\int_{\mathbf{R}^3} |v-u_G|\,\frac{dG(v)}{\|G\|_{\mathscr{B}_0}}\right)^{-1/30} \\
&\quad \times \Omega_B^*((e^{-\frac{1}{2}\beta a t} + J_{G/\|G\|_{\mathscr{B}_0}}(e^{-\frac{1}{2}\beta a t}))^{1/60}).
\end{aligned} \tag{6.12}$$

Further, using the above estimates (6.5), (6.7), (6.4), (6.3), and definition of D_F we compute

$$(1+|u_G|)^3\,(1+T_G)^2 \leqslant \left(1 + \frac{3}{2}\frac{\|F\|_{\mathscr{B}_2}}{\|F\|_{\mathscr{B}_0}}\right)^3 \left(1 + \frac{3}{2}T_F\right)^2;$$

$$\left|\int_{\mathbf{R}^3} |v-u_G|\,\frac{dG(v)}{\|G\|_{\mathscr{B}_0}} - \int_{\mathbf{R}^3} |v-u_F|\,\frac{dF(v)}{\|F\|_{\mathscr{B}_0}}\right| \leqslant \frac{1}{2}\int_{\mathbf{R}^3} |v-u_F|\,\frac{dF(v)}{\|F\|_{\mathscr{B}_0}}.$$

Thus

$$\int_{\mathbf{R}^3} |v-u_G|\,\frac{dG(v)}{\|G\|_{\mathscr{B}_0}} \geqslant \frac{1}{2}\int_{\mathbf{R}^3} |v-u_F|\,\frac{dF(v)}{\|F\|_{\mathscr{B}_0}},$$

and so

$$\begin{aligned}
&(1+|u_G|)^3\,(1+T_G)^2 \left(\int_{\mathbf{R}^3} |v-u_G|\,\frac{dG(v)}{\|G\|_{\mathscr{B}_0}}\right)^{-1/30} \\
&\quad \leqslant 2^{1/30}\left(1 + \frac{3}{2}\frac{\|F\|_{\mathscr{B}_2}}{\|F\|_{\mathscr{B}_0}}\right)^3 \left(1 + \frac{3}{2}T_F\right)^2 (\|F\|_{\mathscr{B}_0})^{1/30} \left(\int_{\mathbf{R}^3} |v-u_F|\,dF(v)\right)^{-1/30}.
\end{aligned} \tag{6.13}$$

Also by definition of the function $J_F(\cdot)$ and using (6.4) we have

$$\begin{aligned}
J_{G/\|G\|_{\mathscr{B}_0}}(e^{-\frac{1}{2}\beta a t}) &= \int_{|v| > e^{\frac{1}{2}\beta a t}} |v|^2\,\frac{dG(v)}{\|G\|_{\mathscr{B}_0}} \\
&\leqslant \frac{4\,\|F\|_{\mathscr{B}_2}}{\|F\|^2_{\mathscr{B}_0}}\,\|F-G\|_{\mathscr{B}_2} + \frac{1}{\|F\|_{\mathscr{B}_0}}\,J_F(e^{-\frac{1}{2}\beta a t}).
\end{aligned} \tag{6.14}$$

Combining (6.11)–(6.14) and using the property $\Omega_B^*(\lambda r) \leqslant (1+\lambda)\, \Omega_B^*(r)$ $(\lambda, r \geqslant 0)$ we get

$$\|g_t - M_G\|_{\mathscr{B}_2} \leqslant C_B C_F \Omega_B^*([e^{-\frac{1}{2}\beta a t} + \|F-G\|_{\mathscr{B}_2} + J_F(e^{-\frac{1}{2}\beta a t})]^{1/60}). \tag{6.15}$$

Next we estimate $\|M_F - M_G\|_{L_2^1}$. We have

$$\begin{aligned}
&\|M_F - M_G\|_{L_2^1}\\
&\leqslant \int_{\mathbf{R}^3} (1+|v|^2)\, \big|\|F\|_{\mathscr{B}_0} - \|G\|_{\mathscr{B}_0}\big|\, (2\pi T_F)^{-3/2}\, e^{-|v-u_F|^2/(2T_F)}\, dv\\
&\quad + \|G\|_{\mathscr{B}_0} \int_{\mathbf{R}^3} (1+|v|^2)\, |(2\pi T_F)^{-3/2}\, e^{-|v-u_F|^2/(2T_F)} - (2\pi T_G)^{-3/2}\, e^{-|v-u_G|^2/(2T_G)}|\, dv
\end{aligned}$$

and for the first term in the right hand side of this inequality we have

$$\begin{aligned}
&\int_{\mathbf{R}^3} (1+|v|^2)\, \big|\|F\|_{\mathscr{B}_0} - \|G\|_{\mathscr{B}_0}\big|\, (2\pi T_F)^{-3/2}\, e^{-|v-u_F|^2/(2T_F)}\, dv\\
&\qquad \leqslant \|F-G\|_{\mathscr{B}_2} (1+|u_F|^2 + 3T_F).
\end{aligned}$$

To estimate the second term, we use the differential mean value formula to the function $(u, T) \mapsto T^{-3/2} e^{-|v-u|^2/(2T)}$, $u \in \mathbf{R}^3$, $T \in (0, \infty)$. Then we obtain

$$\begin{aligned}
&|T_F^{-3/2} e^{-|v-u_F|^2/(2T_F)} - T_G^{-3/2} e^{-|v-u_G|^2/(2T_G)}|\\
&\quad \leqslant \int_0^1 T(\theta)^{-5/2}\, e^{-|v-u(\theta)|^2/(2T(\theta))}\, |v-u(\theta)|\, |u_F - u_G|\, d\theta\\
&\qquad + \tfrac{1}{2} \int_0^1 T(\theta)^{-7/2}\, e^{-|v-u(\theta)|^2/(2T(\theta))} (|v-u(\theta)|^2 + 3T(\theta))\, |T_F - T_G|\, d\theta
\end{aligned}$$

where $u(\theta) = \theta u_F + (1-\theta)\, u_G$ and $T(\theta) = \theta T_F + (1-\theta)\, T_G$. This gives

$$\begin{aligned}
&\int_{\mathbf{R}^3} (1+|v|^2)\, |(2\pi T_F)^{-3/2}\, e^{-|v-u_F|^2/(2T_F)} - (2\pi T_G)^{-3/2}\, e^{-|v-u_G|^2/(2T_G)}|\, dv\\
&\quad \leqslant C_0 \int_0^1 (1+T(\theta)+|u(\theta)|^2)(T(\theta)^{-1/2}\, |u_F - u_G| + T(\theta)^{-1}\, |T_F - T_G|)\, d\theta.
\end{aligned}$$

Also, from the estimates (6.7) and (6.5) we have

$$\frac{1}{2}T_F \leqslant \min\{T_F, T_G\} \leqslant T(\theta) \leqslant \max\{T_F, T_G\} \leqslant \frac{3}{2}T_F$$

$$|u(\theta)| \leqslant \max\{|u_F|, |u_G|\} \leqslant \frac{3}{2}\frac{\|F\|_{\mathscr{B}_2}}{\|F\|_{\mathscr{B}_0}}.$$

Thus

$$\int_{\mathbf{R}^3} (1+|v|^2)\, |(2\pi T_F)^{-3/2}\, e^{-|v-u_F|^2/(2T_F)} - (2\pi T_G)^{-3/2}\, e^{-|v-u_G|^2/(2T_G)}|\, dv$$

$$\leqslant C_0(1+T_F+(\|F\|_{\mathscr{B}_2}/\|F\|_{\mathscr{B}_0})^2)(T_F^{-1/2}\, |u_F-u_G|+T_F^{-1}\, |T_F-T_G|),$$

and combining the estimates (6.5) and (6.6) for $|u_F-u_G|$ and $|T_F-T_G|$,

$$\|M_F-M_G\|_{L^1_2} \leqslant C_F\, \|F-G\|_{\mathscr{B}_2}. \tag{6.16}$$

Now choose $t_0 > 0$ such that

$$\|F-G\|_{\mathscr{B}_2}\, e^{5\,\|F\|_{\mathscr{B}_2}\, t_0} = e^{-(\beta\, \|F\|_{\mathscr{B}_0}/120)\, t_0}$$

$$\text{i.e.}\quad t_0 = \frac{1}{5\,\|F\|_{\mathscr{B}_2} + \frac{\beta\, \|F\|_{\mathscr{B}_0}}{120}} \log\left(\frac{1}{\|F-G\|_{\mathscr{B}_2}}\right) \quad (>0).$$

Then for all $t \in [0, t_0]$, we have, using (6.8),

$$\|f_t-g_t\|_{\mathscr{B}_2} \leqslant \|F-G\|_{\mathscr{B}_2}\, e^{5\,\|F\|_{\mathscr{B}_2}\, t} \leqslant \|F-G\|_{\mathscr{B}_2}\, e^{5\,\|F\|_{\mathscr{B}_2}\, t_0}$$

$$= [\|F-G\|^{\alpha}_{\mathscr{B}_2}]^{1/60} \leqslant 3\pi\Omega^*_B([\|F-G\|^{\alpha}_{\mathscr{B}_2} + J_F(\|F-G\|^{\alpha}_{\mathscr{B}_2})]^{1/60}) \tag{6.17}$$

where $\alpha > 0$ is the constant given in the theorem. In the following we assume that $t \in [t_0, \infty)$. In this case we use the estimate (6.9). Since $a = \|F\|_{\mathscr{B}_0}$ and

$$e^{-\frac{1}{2}\beta a t} \leqslant e^{-\frac{1}{2}\beta\, \|F\|_{\mathscr{B}_0}\, t_0} = (e^{-(\beta\, \|F\|_{\mathscr{B}_0}/120)\, t_0})^{60} = \|F-G\|^{\alpha}_{\mathscr{B}_2}, \qquad t \geqslant t_0$$

it follows from (6.10) and using the property of $\Omega^*_B(r)$ again that

$$\|f_t-M_F\|_{\mathscr{B}_2} \leqslant C_B C_F \Omega^*_B([\|F-G\|^{\alpha}_{\mathscr{B}_2} + J_F(\|F-G\|^{\alpha}_{\mathscr{B}_2})]^{1/60}),$$

and for $\|g_t - M_F\|_{\mathscr{B}_2}$, notice also that $0 < \|F-G\|_{\mathscr{B}_2} < 1$ and $0 < \alpha < \beta \|F\|_{\mathscr{B}_0} / (10 \|F\|_{\mathscr{B}_2}) < 1$, we have by (6.15) that

$$\|g_t - M_F\|_{\mathscr{B}_2} \leqslant C_B C_F \Omega_B^*([\|F-G\|_{\mathscr{B}_2}^{\alpha} + J_F(\|F-G\|_{\mathscr{B}_2}^{\alpha})]^{1/60}).$$

Together these with (6.9) and (6.16) gives for all $t \in [t_0, \infty)$

$$\|f_t - g_t\|_{\mathscr{B}_2} \leqslant C_B C_F \{\|F-G\|_{\mathscr{B}_2} + \Omega_B^*([\|F-G\|_{\mathscr{B}_2}^{\alpha} + J_F(\|F-G\|_{\mathscr{B}_2}^{\alpha})]^{1/60})\}.$$

Combining this with the same kind estimate (6.17) on $[0, t_0]$ gives the global estimate (1.26) in the theorem, and from the above derivation one sees that the constant C_F for defining the function $\Psi_{B,F}(\cdot)$ in the theorem can be written as an explicit and continuous function of $(|u_F|, T_F, \|F\|_{\mathscr{B}_0}, \int_{\mathbf{R}^3} |v-u_F| \, dF(v))$. The proof is completed. ∎

We conclude this paper with the proof of the Corollary to Theorem 3. The strategy is to approximate general initial data F in the $\mathscr{B}_2$ norm by initial data F_R that has finite fourth moments. It is well known that a bound on fourth moments is propagated uniformly in time. By the global stability in $\mathscr{B}_2$ that has just been proved, the "energy tails" of the solution starting from F cannot be too much worse than those of the solution starting from F_R. Since these are controlled by the fourth moment, we obtain an estimate on the "energy tails" of the solution starting from F.

Although property of moment propagation for the Maxwellian model is not new (refs. 14 and 19), we prove a simple explicit bound here that is in a suitable form for our purposes.

Lemma 6.1. Let $0 \leqslant F \in \mathscr{B}_4(\mathbf{R}^3)$, and let f_t be the unique solution of Eq. (1.19) in $C([0, \infty); \mathscr{B}_2(\mathbf{R}^3, \|\cdot\|_{\mathscr{B}_2}))$ with $f_t|_{t=0} = F$. Then

$$\int_{\mathbf{R}^3} |v|^4 \, df_t(v) \leqslant (2 - e^{-At}) \int_{\mathbf{R}^3} |v|^4 \, dF(v), \quad t \geqslant 0 \tag{6.18}$$

where

$$A = \pi \int_0^{\pi} B(\cos(\theta)) \sin^3(\theta) \, d\theta \, \|F\|_{\mathscr{B}_0}.$$

Proof. The global boundedness of the fourth moments is well knowm. To obtain the explicit estimate (6.18), we use the following representation of $|v'|^2$ and $|v'_*|^2$:

$$|v'|^2 = |v|^2 \cos^2(\theta/2) + |v_*|^2 \sin^2(\theta/2) + \sqrt{|v|^2 \, |v_*|^2 - \langle v, v_* \rangle^2} \sin(\theta) \cos(\phi - \alpha),$$
$$|v'_*|^2 = |v|^2 \sin^2(\theta/2) + |v_*|^2 \cos^2(\theta/2) - \sqrt{|v|^2 \, |v_*|^2 - \langle v, v_* \rangle^2} \sin(\theta) \cos(\phi - \alpha),$$

where $\theta = \arccos(\langle v-v_*, \sigma\rangle / |v-v_*|)$ (for $v=v_*$ we define $\theta=0$), $\phi \in [0, 2\pi]$ and α is independent of ϕ. This representation results from the following parameterization of the unit sphere:

$$\sigma = \cos(\theta)\frac{v-v_*}{|v-v_*|} + \sin(\theta)(\cos(\phi)\,\mathbf{i} + \sin(\phi)\,\mathbf{j}).$$

where $\{\frac{v-v_*}{|v-v_*|}, \mathbf{i}, \mathbf{j}\}$ is an orthonormal base of $\mathbf{R}^3$. Using the above representation and recalling that $2\pi \int_0^\pi B(\cos(\theta))\sin(\theta)\,d\theta = 1$ one computes

$$\int_{\mathbf{S}^2} B\left(\left\langle \frac{v-v_*}{|v-v_*|}, \sigma\right\rangle\right) |v'|^4\,d\sigma = \left(\frac{1}{2}-\frac{a}{4}\right)(|v|^4+|v_*|^4) + a\,|v|^2\,|v_*|^2 - \frac{a}{2}\langle v, v_*\rangle^2$$

where

$$a = 2\pi \int_0^\pi B(\cos(\theta))\sin^3(\theta)\,d\theta.$$

From this equality and the local boundedness we have (using conservation law (1.18))

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbf{R}^3} |v|^4\,df_t(v) &= \int_{\mathbf{R}^3} |v|^4\,d(f_t \circ f_t)(v) - \|F\|_{\mathscr{B}_0}\int_{\mathbf{R}^3} |v|^4\,df_t(v) \\
&= -\frac{a}{2}\|F\|_{\mathscr{B}_0}\int_{\mathbf{R}^3} |v|^4\,df_t(v) + a\left(\int_{\mathbf{R}^3} |v|^2\,dF(v)\right)^2 \\
&\quad -\frac{a}{2}\iint_{\mathbf{R}^3\times\mathbf{R}^3} \langle v, v_*\rangle^2\,df_t(v_*)\,df_t(v).
\end{aligned}$$

Neglecting the last term and using the inequality

$$\left(\int_{\mathbf{R}^3} |v|^2\,dF(v)\right)^2 \leqslant \|F\|_{\mathscr{B}_0}\int_{\mathbf{R}^3} |v|^4\,dF(v)$$

we obtain

$$\frac{d}{dt}\int_{\mathbf{R}^3} |v|^4\,df_t(v) \leqslant -\frac{a}{2}\|F\|_{\mathscr{B}_0}\int_{\mathbf{R}^3} |v|^4\,df_t(v) + a\,\|F\|_{\mathscr{B}_0}\int_{\mathbf{R}^3} |v|^4\,dF(v), \quad t \geqslant 0$$

which gives the estimate (6.18) by Gronwall lemma. ∎

Proof of Corollary to Theorem 3. Let $R>0$. By $\int_{|v|>R}|v|^2\,df_t(v) \leqslant \|F\|_{\mathscr{B}_2}$ we can assume that $R\geqslant 1$. Let F_R be a positive measure defined by

$$dF_R(v)=1_{\{|v|\leqslant\sqrt{R}\}}\,dF(v)$$

Then, since $R\geqslant 1$,

$$\|F-F_R\|_{\mathscr{B}_2}\leqslant 2J_F\left(\frac{1}{\sqrt{R}}\right).$$

Now let $t\mapsto f_{R,t}$ be the distributional solution of the Boltzmann equation (1.19) in $C([0,\infty);\mathscr{B}_2(\mathbf{R}^3,\|\cdot\|_{\mathscr{B}_2})$ with $f_{R,t}|_{t=0}=F_R$. Then by Theorem 3 we have

$$\sup_{t\geqslant 0}\|f_t-f_{R,t}\|_{\mathscr{B}_2}\leqslant \Psi_{B,F}(\|F-F_R\|_{\mathscr{B}_2})\leqslant \Psi_{B,F}\left(2J_F\left(\frac{1}{\sqrt{R}}\right)\right).$$

On the other hand, applying Lemma 6.1 to the solution $f_{R,t}$ we have

$$\int_{\mathbf{R}^3}|v|^4\,df_{R,t}(v)\leqslant 2\int_{\mathbf{R}^3}|v|^4\,dF_R(v)=2\int_{|v|\leqslant\sqrt{R}}|v|^4\,dF(v)\leqslant 2R\,\|F\|_{\mathscr{B}_2}.$$

Therefore for all $t\geqslant 0$

$$\begin{aligned}\int_{|v|>R}|v|^2\,df_t(v)&\leqslant \|f_t-f_{R,t}\|_{\mathscr{B}_2}+\frac{1}{R^2}\int_{\mathbf{R}^3}|v|^4\,1_{\{|v|>R\}}\,df_{R,t}(v)\\&\leqslant \Psi_{B,F}\left(2J_F\left(\frac{1}{\sqrt{R}}\right)\right)+2\,\|F\|_{\mathscr{B}_2}\,\frac{1}{R}.\end{aligned}$$

This proves the Corollary. ∎

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